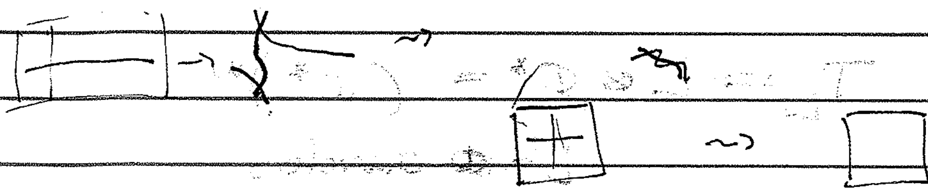


<u>HMS for non-Fano toric "varieties"</u>

joint w/ Katzarkov, Kerr, Diemer, Ballard

$\Sigma \subseteq N$ - simplicial fan, $N \cong \mathbb{Z}^n$

X - associated toric DM stack.

<u>Def'n</u>: X is Fano

$\iff \omega_{\mathbb{P}_\Sigma}^{-1}$ is ample $\iff$ primitive ray generators are all vertices of convex hull.

<u>Def'n</u>: X is nef Fano $\iff \omega_{\mathbb{P}_\Sigma}^{-1}$ is big & nef

$\iff$ the primitive ray generators are on the boundary of the convex hull.

The mirror to X will be a pair $(M \otimes \mathbb{C}^*, W)$

$$M = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}), \quad \underset{\substack{\parallel \\ Y}}{M \otimes \mathbb{C}^* = (\mathbb{C}^*)^n} \xrightarrow{W} \mathbb{C}$$

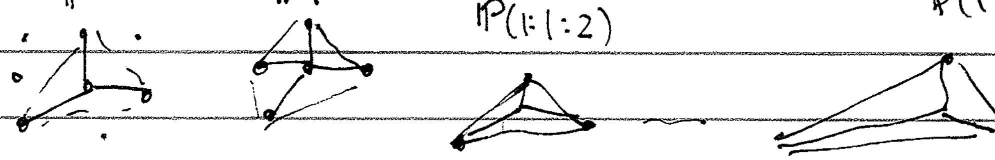
$$W = \sum_{\substack{\alpha \in \Sigma(1) \cup \{0\} \\ c_\alpha \neq 0}} c_\alpha z^\alpha$$

<u>Thm</u> (Abouzaid): X ~~Fano~~ smooth variety, there is a fully faithful functor

$$D^b(X) \hookrightarrow FS(\underbrace{Y, W}_{\text{mirror}})$$

<u>Ex</u>: $\mathbb{P}^2$ $\quad \mathbb{F}_1$ $\quad \mathbb{P}(1:1:2)$ $\quad \mathbb{P}(1:1:3)$

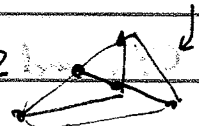
Fano

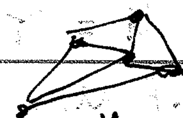


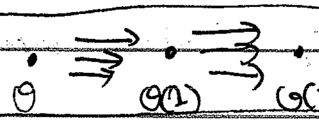
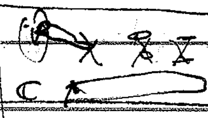
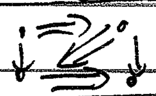
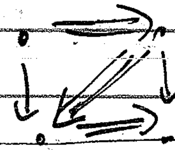
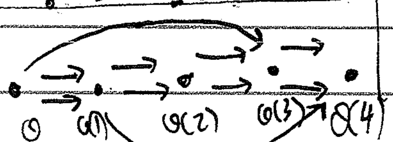
Fano $\mathbb{P}^2$ $\mathbb{F}_1$ $\mathbb{P}(1;1;2)$ $\mathbb{P}(1;1;3)$

T $1+x+y+\frac{1}{xy}$ $\quad 1+x+y+\frac{1}{x}+\frac{1}{xy}$ $\quad 1+x+y+\frac{1}{x^2y}$ $\quad 1+x+y+\frac{1}{x^3y}$

y ↑ → x $a=2$. $\quad 1+x+y+\frac{1}{x}+\frac{1}{x^2y}$ $\quad 1+x+y+\frac{1}{x}+\frac{1}{x^3y}$

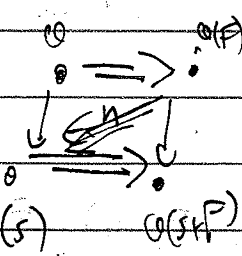
$\mathbb{F}_2$ 
not-Fano

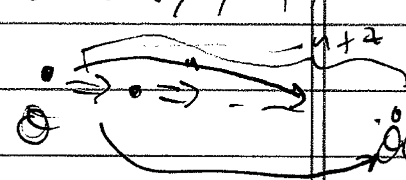
$\mathbb{F}_3$ 
neither

	vertices	critical points of the mirror
$\mathbb{P}^2$		 3
$\mathbb{F}_2$	 + relations.	4
$\mathbb{F}_3$	 + relations	?? 5?
$\mathbb{P}(1;1;3)$		5.

doesn't seem to match!

F^3 is like $\mathbb{P}(1,1,3)$ w/ middle crossed out.

In general, $D^b(\mathbb{F}_n^a)$ 

$D^b(\mathbb{P}(1,1,n))$ 

Mirrors: $\mathbb{F}_n$: $n+2$ crit. values for both!

cross out all middle (w/o degree n maps)

by Abouzaid, So it embeds, but there are more critical points,

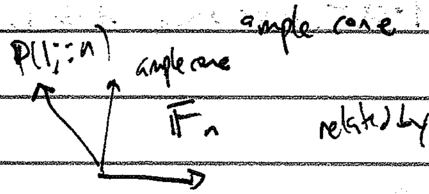
(point: $\mathbb{F}_3$ & $\mathbb{P}(1;1;3)$ have same convex hull, relates birationally.

$D^b(\mathbb{F}_n) \hookrightarrow D^b(\mathbb{P}(1,1,n))$

AKO

(Auroux-Katzarkov-Orlov)

pseudo-effective cone of $\mathbb{F}_n$



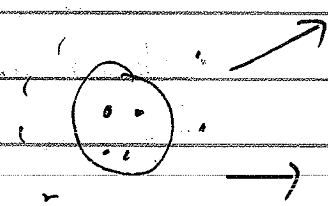
related by wall crossings

deforming Kähler str.

$1+x+y+\dfrac{1}{x}+\dfrac{b}{x^n y}$

$b \to 0$

get mirror to $\mathbb{P}(1,1,n)$, and they're isotopic, so "mirror to $\mathbb{F}_n$ is this category."



$b \to 0$

crit. pts.

4 go to ∞ much slower than $n-2$ critical points.

deforming complex structure

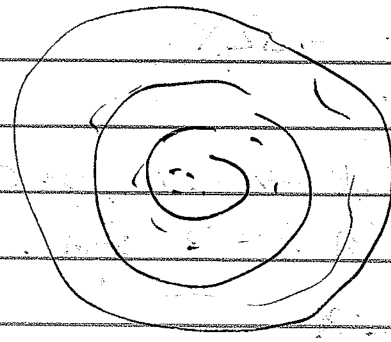
as you pass through walls, 4 crit. pts. distinguish themselves.

Thm: (~~BFK~~ Kawamata, BFK):

~~Thm~~: Toric minimal model runs ~~give~~ for X give semiorthogonal decompositions of $D^b(X)$

Thm (Kerr, Diemer, Katzarkov):

Minimal model runs correspond to maximally degenerated Landau–Ginzburg mirrors.



Break critical points into annuli of different asymptotics as one goes to infinity

Gives semi-orthogonal decomp $\rightsquigarrow$ by ordering induced by annuli.

X' not Fano.

Fact: For any X, there exists a minimal model run such that

$$D^b(X) \hookrightarrow D^b(X')$$

General setup: $G \circlearrowright X$ What is X/G?

Mumford GIT: says "choose L" L is a G-equiv. line bundle

Can define

$$X^{ss}(L) = \{x \in X \mid \exists f \in H^0(X, L^n)^G,\ f(x) \neq 0,\ X_f \text{ affine}\}$$

$$X^{us}(L) = X \setminus X^{ss}. \qquad X /\!/ L := [X^{ss}(L)/G] \text{ global quotient.}$$

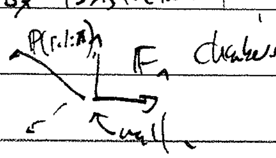
Natural map from $X \to \mathrm{Proj}(\text{coord ring})$

(usual definition is coarse moduli space of this.)

Pic^G parametrizes GIT line quotients.

different GIT quotients give cones in $\mathrm{Pic}^G(X)$,

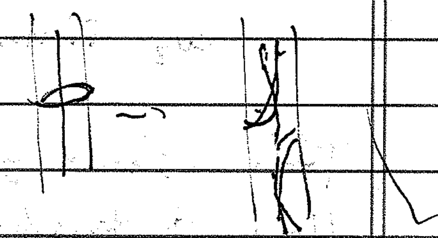
$\rightsquigarrow$ GIT-Fan. $\mathbb{F}_n$ (Cox construction)



SOD: (semi-orthog. decomp.)

$\mathcal{A}_1, \dots, \mathcal{A}_s \subseteq \mathcal{T}$.

$\mathrm{Hom}(\mathcal{A}_j, \mathcal{A}_i) = 0 \quad j > i$.

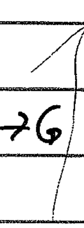


<u>Thm</u> (BFK, Halpern-Leistner)

Let X, Y be neighbors (sep. by wall). $\mu = \langle -k_X, T \rangle$

choose character $T : \mathbb{C}^\times \to G$

$\langle x, T \rangle = 1$



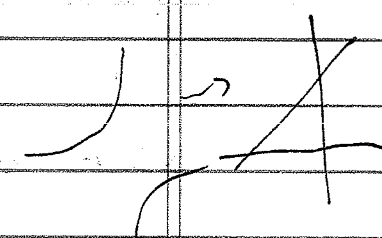
$\exists$ fully faithful functors

$$D^b(X) = \langle D^b(Y), D^b(Z), \dots, D^b(Z)_{(\mu-1)k} \rangle. \quad \mu > 0.$$

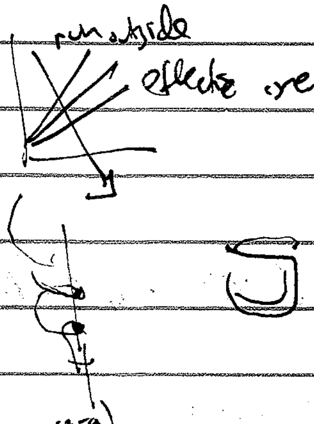
Z is GIT quotient of the fixed locus of T.

(If flop, $\mu = 0$, and have equivalence)

$\mu < 0$: reverse X & Y.



By induction, minimal model runs, followed by minimal model runs on Z for each wall crossing, etc...



run outside effective cone

~~give an exc~~

I get pts in the end for Z.

when you cross boundary, $D^b(Y)$ replaced by zero, & get wall contribution.)

$\simeq \langle D^b(pt), \dots, D^b(pt) \rangle$ vertices.

$(z-a)$

$(z-a)\mathbb{C}[z,z^{-1}] \to \mathbb{C}[z,z^{-1}] \to \mathbb{C}[z,z^{-1}]/(z-a)$