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Beilinson–Kato conjecture

X smooth, proper

~~$K^\cdot(X)$ cohomology theories.~~

X/k k a number field. summed over twistings

Then $K^\cdot(X) \xrightarrow{ch} \oplus H_{et}(X, \mathbb{Q}_\ell^{(\cdot)})$

Conj: this should be an iso after ~~at~~ some changes: completing, inverting, smoothing

$\bar{X} = X \otimes \bar{k}$ alg closure over k, more geometric.

$H^\cdot(Gal(\bar{k}/k), H^\cdot_{et}(\bar{X}, \mathbb{Q}_\ell(\cdot))) \Rightarrow H^\cdot_{et}(X, \mathbb{Q}(\cdot))$

Beilinson conjectures:

$X/\mathbb{Q}$ ~~X/k~~ $H^\cdot_{DR}((X \otimes_{\mathbb{Q}} \mathbb{C})_{an})$ replaces $H^\cdot_{et}(\bar{X}, \cdot)$

it carries an $\mathbb{R}$-MHS, plus an involution $i : H_{DR}(\cdot)$ mixed hodge str.

Def: $H^i_{AH}(X \otimes \mathbb{C}, \mathbb{R}(j))$ Absolute Hodge.

$= R\mathrm{Hom}_{\mathbb{R}MHS}(\mathbb{R}(0), H_{DR}(X \otimes \mathbb{C})_{an}(j))^i$ invariants w.r.t. involution,

cat. of $\mathbb{R}$-MHS.

We have a regulator map

$$K^i(X) \otimes \mathbb{R} \longrightarrow \bigoplus_j H^{2j-i}_{AH}(X \otimes \mathbb{C}, \mathbb{R}(j))^i$$ (invariants)

Conjecture 0: is an iso, but obviously false. (not even for a point $X = \operatorname{Spec} k$)

Correction: $X = \mathrm{Spec}\ k$, assume k has r_1 real embeddings, r_2 complex embeddings

$K^i(X) \otimes \mathbb{R}$			$H^\cdot$
	$i=0$	$\mathbb{R}$ already huge for $\mathbb{Q}$	$\mathbb{R}^{r_1+r_2}$
	1	$K^* \otimes \mathbb{R}$	$\mathbb{R}^{r_1+r_2}$
	2	0	0
	$2i$	0	0
	$4i+1$	$\mathbb{R}^{r_1+r_2}$	$\mathbb{R}^{r_1+r_2}$
	$4i+3$	$\mathbb{R}^{r_2}$.	$\mathbb{R}^{r_2}$

V $\mathbb{R}$-MHS;

$$\begin{matrix} (F^\circ \cap W_0) V_{\mathbb{C}} \\ \oplus \\ W_0 V_{\mathbb{R}} \end{matrix} \longrightarrow W_0 V_{\mathbb{C}}$$

$H_{DR}(\mathrm{Spec}(k \otimes \mathbb{C})) = \mathbb{R}(0)^{r_1+2r_2}$

two eigenspaces: of i: dim r_1+r_2

one: dim r_2.

So, $H^i_{DR}(\mathrm{Spec}\, k \otimes_{\mathbb{Q}} \mathbb{C}(j))^i = \begin{cases} \mathbb{R}^{r_1+r_2} & \text{even}; \\ \mathbb{R}^{r_2} & \text{odd}; \end{cases}$

If: $\mathcal{O}_k \subset k$ the ring of integers,

$X_0 = \mathrm{Spec}\ \mathcal{O}_k$ then

$K^i(X) \otimes \mathbb{R}$

0	$\mathbb{R}$
1	$\mathbb{R}^{r_1+r_2-1}$

> still disagreement, but sum is ½ sum of $H^\cdot$ 0 & 1.

~~$\mathbb{Z}_1$~~ Forget the weight filtration!

$\mathbb{R}$-HS $(V_{\mathbb{R}}, F^\cdot \text{ on } V_{\mathbb{C}}.)$, not abelian, but it's ok.

an involution i.

RHom there is

$$\begin{matrix} F^\circ V_{\mathbb{C}} \\ \oplus \\ V_{\mathbb{R}} \end{matrix} \longrightarrow V_{\mathbb{C}}.$$

Result: $H^{\cdot}_{\mathcal{D}}(X_{an}, \mathbb{R}(\cdot)) \simeq R\mathrm{Hom}_{\mathbb{R}\text{-}HS}(\mathbb{R}(0), \mathcal{H}^{\cdot}_{DR}(X))$.

$\uparrow$ Deligne cohomology (not def'n Deligne used).

~~Deligne do this before taking cohomology~~

$$0 \to \mathbb{R}, \to \underbrace{\mathcal{O}_X \to \Omega^1_X \to \cdots \Omega^i_X}_{} \to 0 \cdots$$

For Deligne cohomology, have non-triviality in negative degrees?

0	$\mathbb{R}^{r_1+r_2}$	$4i$	$\mathbb{R}^{r_1+r_2}$
-1	$\mathbb{R}^{r_1+r_2}$	$2(2i+1)$ ~~$4i+3$~~	$\mathbb{R}^{r_2}$.
$-2i$	0	$2i+1$	0
$-4i+1$	$\mathbb{R}^{r_1+r_2}$		
$-4i+3$	$\mathbb{R}^{r_2}$.		

Conjecture: $X/\mathbb{Z}$ regular

Then we have an exact triangle

$$K^i(X) \otimes \mathbb{R} \longrightarrow \bigoplus_j H^{2j-1}_{\mathcal{D}}((X \otimes \mathbb{C})_{an}, \mathbb{R}(j))$$

$$\longrightarrow (K^i(X) \otimes \mathbb{R})^*[1] \longrightarrow$$

In 0, 1:

$$\mathbb{R} \to \mathbb{R}^{r_1+r_2} \xrightarrow{0} \mathbb{R}^{r_1+r_2-1} |^*$$

$$\mathbb{R}^{r_1+r_2-1} \to \mathbb{R}^{r_1+r_2} \to \mathbb{R}$$

Rotating, get

$$(K^\cdot(X)\otimes\mathbb{R})^* \to (K^\cdot(X)\otimes\mathbb{R}) \to \bigoplus_j H_{\mathcal{D}}(\text{--})$$

can be only non-zero in degree 0, but in deg. 0 can be non-trivial,
in fact in nontrivial cases, gives the height pairing.
(e.g. when $X = E^1_{\mathbb{R}}$).

can't expect to get all of K from $H_{\mathcal{D}}$
b/c K^0 – ~~depends~~ rat'l points of $E_{\mathbb{R}}$ hugely interesting,
depends on curve, & $H_{\mathcal{D}}$ does not.
$\Rightarrow$ doesn't tell us interesting part ~~of~~ K^0.

Noncommutative setting:

$A^\cdot$ DG category $/\mathbb{Z}$ (maybe weaker than "smooth & proper over $\mathbb{Z}$" too strong).

Have: $HP_\cdot(A^\cdot)$ – periodic cyclic homology
(serves as a replacement for D-R cohomology)

~~Thm~~ (HKR): If $D(A^\cdot) = D(X)$ X smooth/proper,
then $HP_i(A^\cdot) = \bigoplus_j H^{2j-i}_{DR}(X)$, lose info, but
(same ~~sum~~ sum as comparison map above)

This has Hodge filtration: inert Bott element. "like Betti realization"

$$HC^-_\cdot(A^\cdot) = F^0\{HP_\cdot(A)\}$$ (subcomplex filtration in a ~~good~~ abstract sense ?)

$\cong HP_\cdot(A^\cdot\otimes\mathbb{C})$

Conjecture (Toën): $K^{st}(A^\cdot\otimes\mathbb{C})\otimes_{\mathbb{Q}[\beta]}\mathbb{C}[\beta,\beta^{-1}]$ $A^\cdot$ homologically smooth & proper,
then take moduli of perfect complexes,

$$\| \quad \pi_i(\mathcal{M}(A^\cdot)_{an})\otimes\mathbb{Q}.$$

st: = semi-topological

moduli of some complexes over A (perfect), derived stack in sense of Toën, has ∞-loop space structure.

Modulo Toën's conjecture,

$HP_\cdot(A \otimes \mathbb{C})$ has a real structure.

Def: $HP_\cdot^{\mathcal{D}}(A \otimes \mathbb{C}) := R\mathrm{Hom}_{\mathbb{R}-HS}(\mathbb{R}(0), HP_\cdot(A \otimes \mathbb{C}))$

Hodge str. w/ filtration

no summation anymore needed, occurs in $HP_\cdot$

Well-known: Regulator map

$$K(A^\cdot) \otimes \mathbb{R} \xrightarrow{r} HP_\cdot^{\mathcal{D}}(A^\cdot \otimes \mathbb{C})$$

For any dga, have $D^{p.f}(A^\cdot)$ perfect cplxes & $D^{pspf}(A)$ pseudoperfect complexes (perfect over base ring, f.d. cohomology)

if A smooth & proper, these are same (Toën)

Can define K theory for either.

Have natural hom pairing

$$D^{pf}(A) \times D^{pspf}(A) \to D^{pf}(\mathbb{Z}).$$

On level of $HP_\cdot(A^\cdot \otimes \mathbb{C})$, induces HP of pseudoperfect complexes

HP of perfect complexes

$$HP_\cdot(A^\cdot \otimes \mathbb{C}) \otimes HP'(A^\cdot \otimes \mathbb{C})$$

$$\downarrow$$

$$\mathbb{R}(1)[2] \quad (\text{or } \mathbb{R}(0) \text{ — can do both b/c both are periodic}).$$

$\boxed{RHom(\mathbb{R}(0), \mathbb{R}(1)[2]) \text{ (cont. ext)} = \mathbb{R}[1]}$

⇝ induces a pairing between

$$HP_\cdot^{\mathcal{D}}(A^\cdot \otimes \mathbb{C})^i \otimes HP'^{\mathcal{D}}(A^\cdot \otimes \mathbb{C})^{?} \to \mathbb{R}[1].$$

Let's say it's non-deg; &

have regulator map, defined for category.

so can look at regulator for pseudo perfect complexes

⇒

pseudoperfect $K'(A^\cdot) \otimes \mathbb{C}^{?} \xrightarrow{r'} HP'^{\mathcal{D}}(A^\cdot \otimes \mathbb{C})^i$

Get:

when dualize, assume f.d.? Actually need since inductive limit in full generality (not sure)

$$K_\cdot(A^\cdot) \otimes \mathbb{R} \xrightarrow{r} HP_\cdot^{\mathcal{D}}(A^\cdot \otimes \mathbb{C})^i \xrightarrow{(r')^*} (K'(A^\cdot) \otimes \mathbb{R})^* \longrightarrow,$$

Conj: This is an exact triangle

Functoriality: If $A^\cdot \to B^\cdot$ get two adjoint functors to dg modules

$Perf A^\cdot \to Perf B^\cdot$

$Pspf B^\cdot \to Pspf A^\cdot$

⇒ $K^\cdot$ covariant, K' contravariant.

⇒ This sequence has correct functoriality.

Now, take A finitely generated projective $\mathbb{Z}$-modules,
(probably too strong).

Tsen's conjecture is very easy to see:
By structure theorem for A,

$$J \subset A \to A^{ss}$$

nilp. radical.

So compare proof for semisimple alg, & show everything behaves well under nilp extension

For A^{ss} easy (bunch of Azumaya algebras).

Assume now $J^2 = 0$, extension;

then Goodwillie: $HP_\cdot(A) = HP_\cdot(A^{ss}/J)$

and $K^{st}(A) = K^{st}(A/J)$, b/c

cohomology of $GL_n(A)$

$GL(A) \to GL(A^{ss})$ unipotent extension, so cohomology same

The main conjecture is also clear.

For semisimple guys, those are Azumaya algs, & we know K theory (by Beilinson's theorem?),

~~For semi~~

For sq. 0 extensions $0 \to J \to A \to A/J \to 0$, $J^2 = 0$.

compute by complex

$$HP\mathcal{D} = F^0 = HC^-(A \otimes \mathbb{C}) \to HP_\cdot(A \otimes \mathbb{C})$$
$$\oplus$$
$$HP_\cdot(A)_{\mathbb{R}}$$

sq. zero: this doesn't change, but this does, is exactly the way $K_1(A')$ changes,

& K' doesn't change, b/c

for K' we have ~~have~~ divisibility, so

$$K'(A) = K'(A/J).$$

Noncommutatively, lose Γ-filtration.