Kapranov

DG-manifolds beyond [0,1]

1. Derived deformation theory (Drinfeld, Kontsevich, —)

All moduli spaces in AG have derived liftings (smooth (singular).

Over $\mathbb{C}$: dg manifolds. $(X = X^0, \mathcal{O}^\bullet_X, d$

smooth variety

$\bigoplus_{i \le 0} \mathcal{O}^i_X$

$\mathcal{O}^\bullet_X$ locally free comm. alg. on $< \infty$ generators

$\mathcal{O}^\bullet_X = \mathcal{O}_{X^0}$,

complex

min. degree of generators

$$x \in X(\mathbb{C}) \quad T^\bullet_x X \in \mathcal{D}^{[0,N]}(\mathrm{Vect})$$

$$\overline{X} = \mathrm{Spec}\, H^0(\mathcal{O}^\bullet_X) \text{ may be sing.}$$

dg stacks: locally quotient dg-mfld / gp.

$$T^\bullet_x N \in \mathcal{D}^{[-1,N]}$$

$$R\mathrm{Bun}_G(Y), \qquad H^\bullet T_{(P)} R\mathrm{Bun}_G(Y) = H^{i+1}(Y, \mathrm{ad}(P))$$

Every moduli space M arises as $\overline{RM}$ $\Rightarrow$ $\exists$ canonical graded algebra $H^\bullet(\mathcal{O}_{RM})$ w/ $H^0 = \mathcal{O}_M$

$\bigoplus_{i \le 0} H^i$

Quasi-smooth case: $[0,1]$ - mfd

$$T^\bullet_x \in \mathcal{D}^{[0,1]} \quad \text{or} \quad (\mathcal{D}^{[-1,1]} \text{ for stacks}$$

e.g. $\mathrm{Spec}(\Lambda^\bullet(E^*), \lrcorner\, ds)$ Koszul complex

In this case, $\underline{H}^i(\mathcal{O}_X) = 0$ for $i \ll 0$.

Then, $\sum (-1)^i [\underline{H}^i] \in K(\bar{X})$ virt. fun. class (Kontsevich)

e.g. $RBun_G$ (surface) is quasi-smooth.
But $RBun_G$ (3-fold) is NOT.

Examples of derived moduli spaces
(a) Commuting varieties: $\mathfrak{g}$ f.d. Lie alg

$$C^r(\mathfrak{g}) := \{(x_1, \dots, x_r) \in \mathfrak{g}^r / [x_i, x_j] = 0\}$$

$$= \underline{Hom}_{Lie}(\mathbb{C}^r, \mathfrak{g}).$$

Derived comm. variety

$$RC^r(\mathfrak{g}) : \text{add to } \mathbb{C}[\mathfrak{g}^r]$$

matrix elts. of x_i

- $\binom{r}{2}$ generators in degree -1 x_{ij}

with $x_{ij} \xrightarrow{d} [x_i, x_j]$

$$[x_i, x_{jk}] - [x_j, x_{ik}] + [x_k, x_{ij}] \xrightarrow{\text{Jacobi}} 0$$

so add
$\binom{r}{3}$ generators x_{ijk} degree 2 mapping here

$\mathfrak{h}$ any finitely presented Lie algebra,

$$\underline{Hom}_{Lie}(\mathfrak{h}, \mathfrak{g}) = \{\text{Lie alg-homomorphisms}\}$$ like extension + Jacobi.

$$R\underline{Hom}_{Lie}(\mathfrak{h}, \mathfrak{g}) = \mathrm{Spec}\, C_{Lie}(C^{\geq 1}_{Lie}(\mathfrak{h}) \otimes \mathfrak{g}).$$

If $\mathcal{L} = \{\mathcal{L}^1 \to \dots \to \mathcal{L}^N\}$ f.d. dg Lie alg in degrees ≥ 1,
then $C_{Lie}(\mathfrak{h})$ comm. dg-alg in degrees ≤ 0..

Koszul duality formula

$$\underbrace{\mathbb{C}[R\underline{\mathrm{Hom}}(h,g)]}_{\text{comm.}}{}^{!} = \underbrace{h^{\overset{\text{Koszul dual?}}{!}} \otimes g}_{\text{Lie}}$$

(6) Varieties of complexes. $\{V^0, V^1, \ldots, V^n\}$.

$V^\bullet$ fin-dim. graded $\mathbb{C}$-vectorspaces.

$Com(V) : \{D_i : V^i \to V^{i+1}) \mid D^2 = 0\}$.

Buchsbaum - Eisenbud - ...

$\prod GL(V^i)$ acts

actions on $\mathbb{C}[Com(V)]$ has simple spectrum

Described by: Some sequences of Young diagrams. $(\alpha_1, \ldots, \alpha_n)$

Many examples this acyl (e.g. complex structure integrability)

Derived version: $R\,Com\,(V)$.

Introduce matrices of generators

$D_{i,i+2} \in Hom(V^i, V^{i+2})$ of degree -1

$$\xrightarrow{d} D_{i+1,i+2} \circ D_{i,i+1}.$$

$D_{i,i+3} \in Hom(V^i, V^{i+3})$ of degree -2

... .

A short definition: graded Lie algebra $\mathcal{L} = \bigoplus_{i<j} Hom(V^i, V^j)$ in degree $j-i$.

$R\,Com(V^\bullet) = Spec\ C^\bullet_{Lie}(\mathcal{L})$.

Functor represented by $R\mathrm{Com}(V)$ on dg-algebras

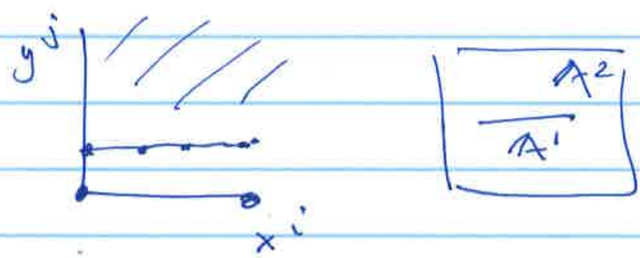
$$A \longmapsto \{ D \text{ on } A \otimes V \text{ s.t. } (D + d_A \otimes 1)^2 = 0 \}.$$

K.-Pimenov: (1) $H^\bullet(\mathcal{O}_{R\mathrm{Com}(V)})$ is ~~not~~ typically not finitely generated as an algebra, also H^{even}.

Two sources (types of phenomena).

Attempt

— contractions by affine manifolds e.g. $\{ f : \mathbb{A}^2 \to \mathbb{C} \mid f|_{\mathbb{A}^1} = \text{const} \}$



$\mathbb{A}^2/\mathbb{A}^1$

— Deletions: $H^\bullet(\mathbb{A}^2 - \{0\}, \mathcal{O})$ $\quad H^0$ ok (functions on $\mathbb{A}^2$)

H^1 not f.g. over H^0.

(2) $H^\bullet(\mathcal{O}_{R\mathrm{Com}(V)})$ still has simple spectrum as $\prod GL(V^i)$

[generalizing B-E].

(3) this is an analogue of Kostant's theorem

$\mathcal{L}$ is nilp. radical of a parabolic subalgebra in a superalgebra of $\mathfrak{gl}(m|n)$ $\quad m = \sum_{i\ \text{even}} \dim V^i$, $n = \sum_{j\ \text{odd}} \dim V^j$.

Kostant: $H^\bullet_{Lie}(\mathfrak{n}, \mathbb{C}) = \bigoplus_{w \in W} \mathbb{C}_{w(\rho)-\rho}$ as T-module $\quad$ of reductive Lie $\mathfrak{b} \supset \mathfrak{n}$. T = max. torus.

weyl gp.

If $V^\bullet = \{\mathbb{C}; 0; \mathbb{C}; 0; \cdots\ 0; \mathbb{C}\}$.

then $\mathcal{L}$ is purely even $\begin{pmatrix} 0 & * \\ 0 & \ddots & 0 \end{pmatrix} \subset \mathfrak{gl}_n$.

③ Applications to $\mathrm{Hilb}(\mathbb{C}^3)$: (K-Vasserot)

$\underset{\text{smooth}}{M} \xrightarrow{f} \mathbb{C}$ $\quad$ $\mathrm{Crit}(f)$ critical locus.

$R\,\mathrm{Crit}(f) = \mathrm{Spec}(\Lambda^\bullet T_M, \lrcorner\, df)$ $[0,1]$ manifold if M manifold.

$[-1, 2]$ -dg stack if M smooth stack.

↑ not quasi-smooth.

Donaldson-Thomas philosophy: Any derived moduli stack associated to a 3-D CY (a $[-1,2]$ dg stack) is locally represented as $R\,\mathrm{Crit}(M, S)$

M = smooth nc moduli stack.

S version of Chern-Simons.

Ex: $\mathfrak{g}$ reductive Lie algebra $\langle\ ,\ \rangle$ inv. product

$\mathrm{Com}^3(\mathfrak{g}) = \mathrm{Crit}\{\mathfrak{g}^3 \to \mathbb{C} \quad (x,y,z) \mapsto \langle x, [y,z]\rangle\}$

$R\,\mathrm{Crit}(S)$ (this S) is a $[0,1]$ mfld.

Cor

$R\,\mathrm{Com}^3(\mathfrak{g})$ is not $[0,1]$ b/c of generators x_{123} in degree -2.

Not the same as (obtained by removing x_{123}).

$RC^3(\mathfrak{gl}_n)/\!/GL_n$

$= R\,\mathrm{Crit}(\mathfrak{gl}_n^3/\!/GL_n \xrightarrow{S} \mathbb{C})$.

K-th = GL_n-equiv. K-theory of f.g. dg-modules.

$\mathrm{Hilb}^n(\mathbb{C}^3) = \{\text{codim. } n \text{ ideals in } \mathbb{C}[t_1,t_2,t_3]\}$ singular scheme.

$T = (\mathbb{C}^*)^3$ acts: fixed pts $\longleftrightarrow$ 3d Young diagrams.

$$\mathrm{Crit}\left(\left\{\begin{matrix}\text{codim. } n \text{ left ideals} \\ \text{in } \mathbb{C}\langle t_1,t_2,t_3\rangle\end{matrix}\right\} \xrightarrow{S} \mathbb{C}\right). \quad S(I) = \text{"}\mathrm{tr}\left(t_1[t_2,t_3]\right)\Big|_{\mathbb{C}\langle t_1,t_2,t_3\rangle / I}\text{"}$$

$\mathrm{DHilb}^n(\mathbb{C}^3) = R\mathrm{Crit}(S)$ $[0,1]$ dg scheme.

$K_T(\mathrm{DHilb}^n)$: has a basis labelled by 3d Young diagrams.

Let $M = \bigoplus K_T(\mathrm{DHilb}^n\, \mathbb{C}^3)$ MacMahon space.

$K_T(\mathrm{pt.}) = \mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}, q_3^{\pm 1}]$

Want to make it a module over something:

Need: non quasi-smooth objects.

$\mathcal{A} = \bigoplus K_{T\times GL_n}$(fin. gen. dg-modules on $RC^3(gl_n)$)

K-theoretic Hall algebra (general formalism + presentation of f.g.)

Thm: 1) M is naturally a $\mathcal{A}$-module.

2) $\mathcal{A}$ = shuffle alg. over defined by kernel fcn.

$$\frac{\prod(1-q_i t)(1-q_1q_2q_3 t)}{(1-t)\prod_{i<j}(1-q_iq_j t)}.$$

Correspondences: $m<n$ versions of flag Hilbert scheme. $\{(I \supset J)\}$

$$\begin{array}{ccc} & \mathrm{DHilb}^{m,n} \ [0,2] \text{ dg-scheme} & \\ \swarrow & & \searrow \\ \left[R\,\mathrm{Com}^3(gl_{n-m}) /\!/ GL_{n-m}\right] \times \mathrm{DHilb}^m & & \mathrm{DHilb}^n \\ [-1,2]\text{-stack} \quad [0,1] & & [0,1] \end{array}$$

$\rho = \square$.

gl_n.