Keel I:

Ex: $U = (\mathbb{C}\setminus\{0\})^N = \{(z_1, \ldots, z_N) \mid z_i \neq 0\}$

$$V = \Gamma(U, \mathcal{O}_U) = \bigoplus_{q \in \mathbb{Z}^n} \mathbb{C}\cdot z^q$$

Prop: This "Basis" (up to scaling) of V is canonical.

Pf: $f \in U$. f nowhere zero $\Longleftrightarrow f = \lambda \cdot z^q$, for some $\lambda \in \mathbb{C}^*$, q.
So "monomials" intrinsic notion.

Conjecture: (GHK): The same thing works for any affine Calabi-Yau U with maximal boundary.

- V has a canonical basis.
- structure constants for multiplication in this basis are determined by counts of rational curves on the mirror.

Def: U^N - smooth, called CY if:

- $\exists\ \omega \in \Gamma(U, (\Lambda^N T_U)^*)$ with ω has, at worst a simple pole along any

$$U \overset{\text{open}}{\subset} Y \supset \text{divisor } D. \qquad \mathrm{val}_D(\omega) \geq -1.$$

- ω with this property is unique up to scaling.
want all vol. forms on U.

e.g. (almost equivalent)

$\Leftarrow$ (Y, D) (projective) simple normal crossing (s.n.c.) comp. of U

and $K_Y + D \equiv 0$.

Then $U = Y \setminus D$ is CY, and (Y, D) "minimal model" of U.

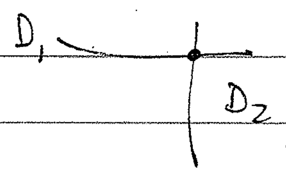
(∞ many equally good).

Non-example: elliptic curve $-$ pt. has loads of volume forms

Actually, Take this as the definition.

Maximal boundary := want D to contain a zero stratum:

$p \in D \subset Y$ looks like $0 \in z_1 \cdot \ldots \cdot z_N = 0 \subset \mathbb{C}^N_{z_1, \ldots, z_N}$.



divisorial valuation　　rational function

Key def'n:

$$U^{trop}(\mathbb{Z}) = \left\{ v : \mathbb{Q}(U) \setminus \{0\} \to \mathbb{Z} \;\middle|\; v(\omega) < 0 \right\} \cup \{0\}$$

$$= \left\{ (E, m) \;\middle|\; \begin{array}{l} E \text{ is a boundary divisor in some minimal model} \\ m > 0 \end{array} \right\} \cup \{0\}.$$

Note: $U \overset{open}{\subset} Y \supset D$. Then get function

$$\mathrm{ord}_D : \mathbb{Q}(U) \setminus \{0\} = \mathbb{Q}(Y) \setminus \{0\} \to \mathbb{Z}$$

$f \mapsto$ order of zero or pole.

$\mathrm{ord}_D(\omega)$ makes sense.

ex. $L \cong \mathbb{Z}^N$. $T_L := L \otimes \mathbb{C}^* = (\mathbb{C}^*)^N$

~~L = characters~~

~~CY w/ maximal boundary~~

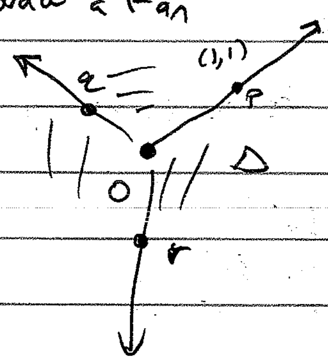
$$\omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

minimal model $\Leftrightarrow$ T_L toric compactification, and

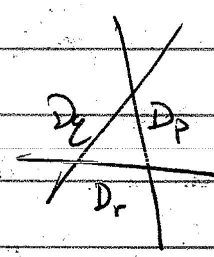
$T_L^{trop}(\mathbb{Z}) = L$: Toric geometry

e.g. $(\mathbb{C}^*)^2 = T_L$ $\quad L = \mathbb{Z}^2$.

"draw a Fan"



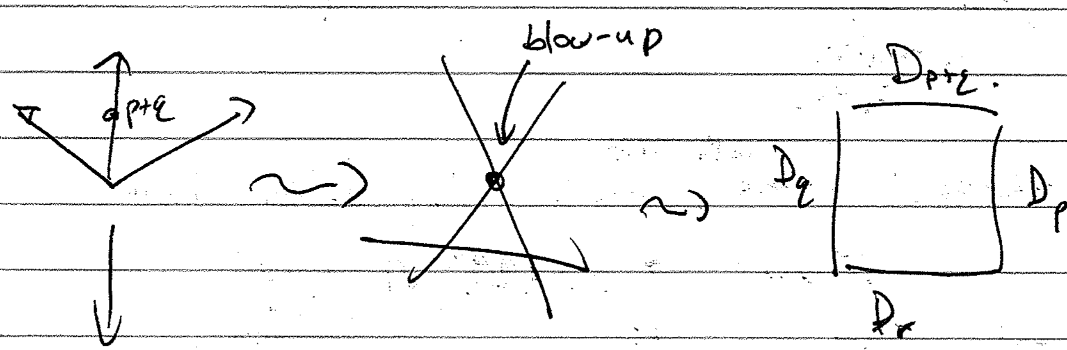
$TV(\Delta) = \mathbb{CP}^2$



Y = board.

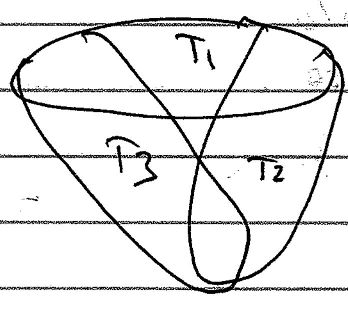
D = 3-lines.

$U = Y \setminus D$



Next example:

$T_1, T_2, \dots$ bunch of tori.



$U = \bigcup T_i$

glue in such a way that the volume forms patch.

So U is automatically CY w/ maximal boundary.

Note: If $T \overset{\text{open}}{\subset} U$ then $\omega_U|_T = \omega_T$.

and of course $Q(T) = Q(U)$

so, $T^{\text{trop}}(\mathbb{Z}) \overset{\text{as set}}{=\!=} U^{\text{trop}}(\mathbb{Z})$.

So if $T = T_L$, this $= L$.

e.g. all cluster varieties occur this way.

e.g. (FZ) Fomin – Zelevinsky G semi-simple

$G/U \to G/B$ Torus bundle w/ fibre $T_{\mathrm{Pic}(G/B)^*}$.

universal torsor $\qquad \| $

Flag manifold.

U unipotent radical.

$\Gamma(G/U, \mathcal{O}) =: \underbrace{\mathrm{Cox}(G/B)}_{\text{cox ring of } G/B}$

$= \bigoplus_{L \in \mathrm{Pic}(G/B)} H^0(G/B, L)$.

$$\Gamma(G/U, \mathcal{O}) = \bigoplus_{L \in \mathrm{Pic}(G/B)} H^0(G/B, L).$$

FZ: G/U has cluster structure, so it's CY w/ maximal boundary (small lie, but almost).

Θ-Functions conjecture says this has a canonical basis.

Rmk: By Borel-Weil-Bott,

all irred. reps. of G

So, we get a basis for every representation of a group, by a method that a priori has nothing to do with G.

In fact, canonical basis is known: ~~L~~ Lusztig ~~C~~ canonical basis $\mathcal{CB}$ in fact, several!)

Goal: Get Lusztig basis without rep. theory.

U CY w/ maximal boundary. $V :=$ the vectorspace w/ basis $U^{trop}(\mathbb{Z})$

$$= \bigoplus_{q \in U^{trop}(\mathbb{Z})} k \cdot \Theta_q .$$

Goal: put a k-algebra structure on V s.t. $\mathrm{Spec}(V)$ is CY & $\mathrm{Spec}(V) =$ mirror.

Thm: This works for U dim. 2 & for lots of cluster varieties, e.g. any cluster variety w/ "acyclic seed."

Note: $0 \in U^{trop}(\mathbb{Z})$ is canonical: you can scale valuations.

ϑ_0 special, so have

$$V \xrightarrow{\text{trace}} k$$

$$f \longmapsto \text{coeff. of } \vartheta_0 .$$

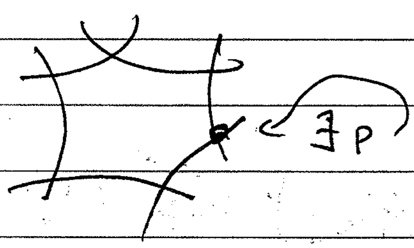
Claim: $\exists$ canonical m-point functions

$$\underbrace{V \times \cdots \times V}_{m} \longrightarrow k$$

claim:

$\exists$ canonical class $\gamma \in H_N(U, \mathbb{Z})$.

def'n: of max. ∂:



s.t. $p \in D \subset Y$ looks like 0 in $z_1 \cdots z_N = 0 \subset \mathbb{C}^N_{z_1 \cdots z_N}$

the locus $|z_1| = |z_2| = \cdots = |z_N| = \varepsilon$

$$\parallel$$

$$\gamma = (S^1)^N \subset U.$$

Claim: $[\gamma] \in H_N$ is canonical (1) given (Y, D) independent of p.
(2) independent of choice of minimal model.

Thm: (1) is true in all dimensions — (uses Shokurov connectedness principle in birational geometry (deep))

(2) True in dim 2 (b/c very restrictive minimal models in dim 2, no flops)

True gives cluster structure.

- We can fix scaling on ω: require $\int_\gamma \omega = 1$.

Now, define

$$\text{Trace} : \Gamma(U, \mathcal{O}_U) \longrightarrow \mathbb{C}$$
$$f \longmapsto \int_{[\gamma]} (f \cdot \omega)$$
top degree, b/c closed.

Multipoint formulas:

$$V \times \cdots \times V \longrightarrow \mathbb{C}$$

$$\langle f_1, \ldots, f_N \rangle = \text{Trace}(f_1 \cdot f_2 \cdots f_N) = \int_\gamma (f_1 \cdots f_N)\omega.$$

Conj (thm in dim 2, almost for clusters).

2-point pairing.

$$\Gamma(U, \mathcal{O}_U) \times \Gamma(U, \mathcal{O}_U) \longrightarrow \mathbb{C} \text{ is}$$

non-degenerate, i.e.

$f \in V$ is determined by functional $\longrightarrow \langle f, - \rangle$.

Note: Now mult. rule on $\Gamma(U, \mathcal{O}_U)$ is determined by trace, $\langle , \rangle$, $\langle , , \rangle$.

e.g. f, g
want to know $f \cdot g$.
Need to know $\langle f \cdot g, - \rangle = \langle f, g, - \rangle$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Trace}(f \cdot g \cdot \_\_)$.

Back to beginning:

$$V = \bigoplus_{q \in U^{trop}(\mathbb{Z})} k \cdot \vartheta_q.$$ To give multiplication, give Trace, + m-point functions

I think: $V = \Gamma(U^\vee, \mathcal{O})$.
$U^\vee$ ↑ mirror

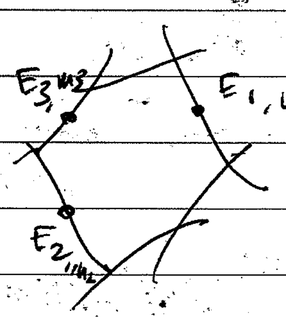
Map is Bilinear; Just need to give

$$\langle \vartheta_{q_1}, \vartheta_{q_2}, \ldots \vartheta_{q_N} \rangle \in K.$$

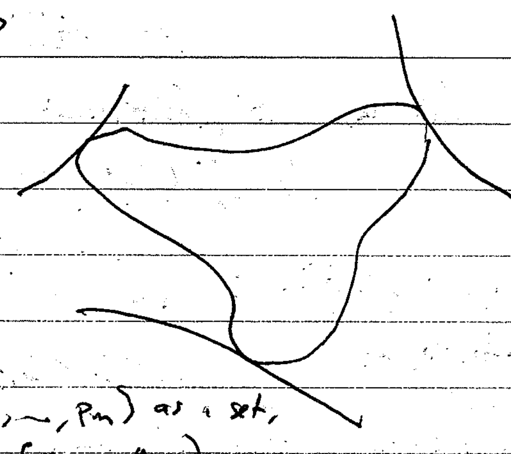
$q_i = (E_i, m_i)$: Easy to put all E_i in same minimal model (by blowing up enough times).

Data given:

(in dim. 2, D is a cycle of $\mathbb{P}^1$'s).



Fix general point $y \in Y$.
Then count rational curves



i.e. maps
$(\mathbb{P}^1, p_1, p_2, \ldots, p_n, x) \xrightarrow{\varphi} Y$
$\varphi(x) = Y$, and $\varphi^{-1}(D) = \{p_1, \ldots, p_n\}$ as a set,
w/ multiplicities $(m_1, \ldots, m_n)$.

Naive dim count:

dim of such maps is $m+1-3 = \dim \mathcal{M}_{0,m+n}$.

If we fix the modulus of $(\mathbb{P}^1, p_1, p_2, \dots, p_n)$

(e.g. $n=3$, cross ratio).

Then, expected dimension $= 0$, so count. (write as GW #).

Slight lie: need to fix homology class of image.

$$\alpha \in H_2(Y, \mathbb{Z}) \text{ of image curve}$$

$$\overset{\shortparallel}{[C]} \in \underset{\substack{\shortparallel \\ \text{Mori cone..}}}{NE(Y)}$$

Given
$(E_1, m_1), \dots, (E_n, m_n) + [C]$ get #

Thus, canonical trace lives in

$$V \times \dots \times V = R = k[NE(Y)]$$

(Novikov ring-type).

Now, define

$$V = \bigoplus_{q \in U^{trop}(\mathbb{Z})} R \cdot \vartheta_q \qquad \overset{\shortparallel}{\bigoplus_{C \in NE(Y)}} k \cdot z^C.$$

Mori cone $NE(Y)$.

Good: want a mirror family anyway.

Conj: Take (Y, D) $K_Y + D = 0$, Assume D supports ample, has 0 stratum.

$(\Rightarrow U = Y \setminus D$ affine $(Y$ w/ max. $\partial)$

and $NE(Y)$ finite polyhedral.

(1) $\exists$ unique R-algebra structure on V s.t.

(1) $\vartheta_0 = 1$ unit.

(2) Coeff of $\underset{\underset{\Theta_0}{\parallel}}{1}$ of $(f_1 f_2 \cdots f_r) = \langle f_1, f_2, \dots, f_r \rangle$

(2) $\mathrm{Spec}(V) \xrightarrow{\text{flat (automatically)}} \mathrm{Spec}(R)$

$$\begin{array}{ccc} & & \mathrm{Spec}(R) \\ & & \parallel \\ \mathcal{U} & & TV(NEF(Y)) \\ & & \cup \\ \mathcal{X} & \xrightarrow{\pi} & T_{\mathrm{Pic}(Y)} \end{array}$$

↑ toric variety given by nef cone of Y.

↑ torus.

Want: π is a family of affine CY N-folds w/ maximal ∂, HMS dual to $\mathcal{U}$.

(3) $V = SH^0(\mathcal{U})$ deg. 0 part of it.

(4) Take $\mathcal{U}'$ (gen.) fibre $\mathcal{U}'$ is affine CY w/ max ∂.

So run program again:

$\mathrm{Spec}(V') \longrightarrow TV(NEF(Y'))$

↑ this is the universal deformation space of the original $\mathcal{U}$.

But now $\mathcal{U}$ has Θ-functions.

(8)

(5) If U is hol. symplectic, i.e. $(\omega = \Lambda^{top} \Omega^2)$.
Then, mirror is canonically $=$ to U
(so don't need to go twice).
(e.g. dim 2, unimodular cluster).

Thm: "This" works for U of dim. 2 and for lots of clusters, e.g. acyclic.

We didn't say what the coeff. of ϑ_r is in $\vartheta_p \cdot \vartheta_q$;
give $\langle \vartheta_p \vartheta_q, \vartheta_s \rangle$ $\forall s$.

We also have a conjecture for this coeff. That is what we actually prove.

$U^{trop}(\mathbb{Z}) \subset U^{trop}(\mathbb{R})$. claim there's a natural way to build real valuations.

All cases we discuss:

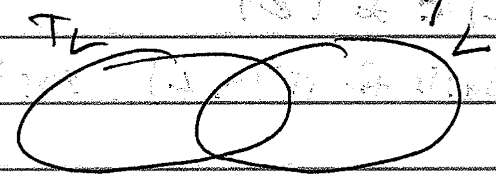
$$U \overset{open}{\supset} T_L, \text{ so}$$

$$= T_L^{trop}(\mathbb{Z}) = L \subset L_{\mathbb{R}} \cong \mathbb{R}^N.$$
$$\overset{\shortparallel}{\mathbb{Z}^N}$$

$\mathcal{B} = U^{trop}(\mathbb{R})$. $\mathcal{B} \underset{homeo.}{\cong} \mathbb{R}^N$
is a piecewise linear identification.
Why?
when, $U \supset T_L$, generally $\exists$ infinitely many other tori, e.g. in particular 2. $T_{L'} \subset U \supset T_L$

So, $U^{trop} = T_L^{trop} = L_{\mathbb{R}}$

$$\begin{array}{c} U^{trop} \\ \| \\ T^{trop}_{L'} \\ \| \\ L'_{\mathbb{R}}. \end{array}$$



$$L' \xleftarrow{\sim} L$$

piecewise linear isomorphism.

So U^{trop} does not have a canonical linear structure, just a canonical PL structure.

(there is a way to give it a lin - str w/ singularities.)