Keel II: L lattice $= \mathbb{Z}^N$ $\quad L$ = characters of T_L

$T_L = L \otimes \mathbb{C}^*$ $\quad L^* =$ ~~char~~ cocharacters of T_L

If $G \in L^*$, then $z^G : T_L \longrightarrow \mathbb{C}^*$

Function, in coordinates is a monomial.

Basic mutation:

Take $v \in L$, $F \in L^*$. want $F(v) = 0$.

$m = m_{(v,F)} : T_L \dashrightarrow T_L$ rational map.

on characters, $m^*(z^G) = z^G\left(1 + z^F\right)^{G(v)}$

Key point: $m^*(\omega) = \omega$.

This induces an isomorphism $m^* : Q \longrightarrow Q$

field of rat'l functions

Get canonically a map

$m^t_{(v,F)} : T_L^{trop} \dashrightarrow T_L^{trop}$ defined as follows:

Given a valuation $v : Q(T_L) \setminus \{0\} \xrightarrow{v} \mathbb{Z}$ $\quad v \mapsto v \circ m^*$

$$\begin{tikzcd} Q(T_L)\setminus\{0\} \arrow[r, "v"] & \mathbb{Z} \\ Q(T_L)\setminus\{0\} \arrow[u, "m^*"] \arrow[ur, "v \circ m^*"'] & \end{tikzcd}$$

Recall, $T_L^{trop} = L$ in this case, and

$m^t : L_{\mathbb{R}} \longrightarrow L_{\mathbb{R}}$ $\quad$ ($L_{\mathbb{R}} = \mathbb{R}^n$) compute:

$x \longmapsto x + F(x)_+ v$

Notation: If $r \in \mathbb{R}$, write $r_+ = \begin{cases} r & \text{if } r \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

$m^t : \mathbb{R}^N \to \mathbb{R}^N$ $\quad$ if $L \cong \mathbb{Z}^2$.

$\quad\;\; = L_{\mathbb{R}} \quad\;\; = L_{\mathbb{R}}$.

$F < 0$	$F \geq 0$
Id	$x \mapsto x + F(x)v$

PL function

Ex: Very similar CY: Let $N = \mathbb{Z}^N$

Take copies of T_N, $T_{N,s}$, $s \in I$.

CY w/ maximal boundary $\to \mathcal{A} = \bigcup_{\substack{s \in I \\ \text{seeds}}} T_{N,s}$; glued by maps

$m_{ss'} : T_{N,s} \dashrightarrow T_{N,s'}$ of this sort.

(careful: might have a cocycle condition. F. G. are very careful about this, & don't have deformations)

e.g. there's a tree of tori.

each seed s gives an identification

$$\underbrace{\mathcal{A}^{trop}(\mathbb{R})}_{\text{piecewise linear manifold.}} = T_{N,s}^{trop}(\mathbb{R}) = N_{\mathbb{R}}$$

Note: These come in dual pairs.

Recall $m_{(v,F)}$ was defined by $v \in L$, $F \in L^*$, w/ $F(v) = 0$

$$\updownarrow$$

$m_{(F,v)}$ thinking of main lattice as L^* & dual of $L^{**} = L$.

~~Get~~ For $m_{(v,F)} : T_{N,s} \dashrightarrow T_{N,s'}$

get $m_{(F,v)} : T_{M,s} \dashrightarrow T_{M,s'}$

=: FG-dual (Fock-Goncharov).

Namely, define $\mathcal{X} := \bigcup_{s \in I} T_{M,s}$

Naive hope: $\mathcal{A}$ and $\mathcal{X}$ are mirror dual.

Ex. hope: $\mathcal{A}^t(\mathbb{Z}) \xleftrightarrow{\text{basis}} \Gamma(\mathcal{X}, \mathcal{O}_X)$

and $\mathcal{X}^t(\mathbb{Z}) \longleftrightarrow \Gamma(\mathcal{A}, \mathcal{O}_X)$.

Thm: This actually works (w/ corrections).

Now, U CY w/ maximal boundary, e.g. $\mathcal{A}$ or $\mathcal{X}$

$$V = \bigoplus_{q \in B(\mathbb{Z})} k \cdot \vartheta_q \qquad B = U^t(\mathbb{R}) \quad (\cong \mathbb{R}^N \text{ in our examples})$$

Describe (conj) mult. rule directly (w/o using Mori cone enlargement)

informally $\vartheta_p \cdot \vartheta_q = \sum_{r \in B(\mathbb{Z})} \alpha(p,q,r)\, \vartheta_r$

• Straight lines in $B(\mathbb{R})$ don't make sense

Main (conj): $\exists$ canonical piecewise straight graphs

could call them "tropical discs", but ~~...~~ not quite right — call "broken lines."

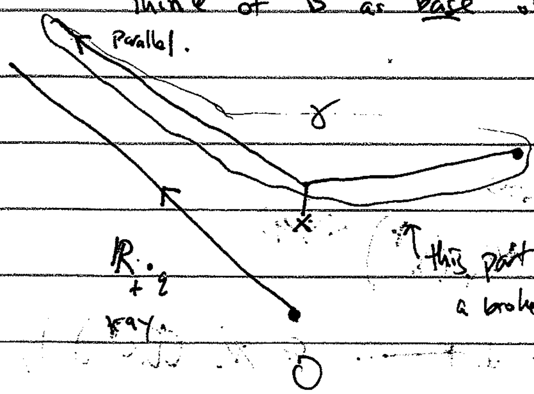
which paths are "broken" • $\mathcal{A}$ or $\mathcal{X}$ case: For each ~~...~~ $N_{s,\mathbb{R}} = B(\mathbb{R})$ (each seed)

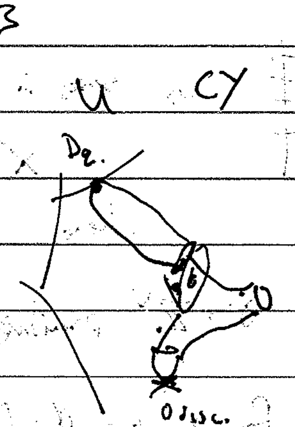
• straight lines in $N_{s,\mathbb{R}}$ are "broken."

• Notion indep. of the piecewise linear mutation

Morally: Tropicalize discs (Maslov 2).

Think of B as base of SYZ fibration: (integer linear manifold, so has $\mathcal{O}$)





U CY $\quad \mathcal{U} \quad T_b$

$\downarrow \quad \downarrow$ fibres $(S^1)^n$.

$B \ni b$

For broken lines, only draw path. Say γ goes "from q" to "b."

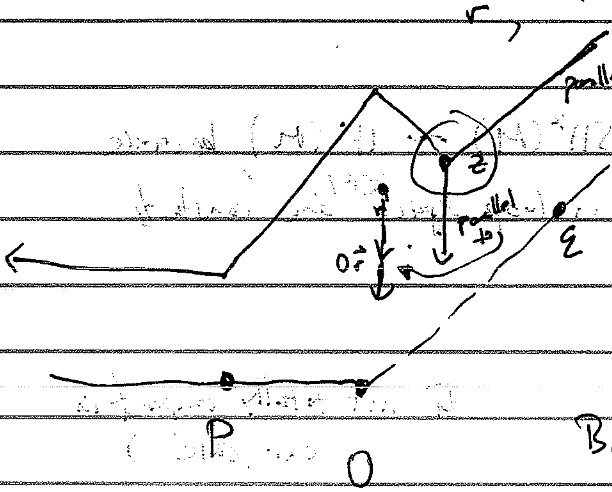
(q is divisor at ∞, kind of!)

Then,

$$\Theta_p \cdot \Theta_q = \sum_r \alpha(p,q,r)\,\Theta_r.$$

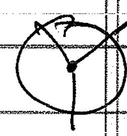
$\alpha(p,q,r)$ counts pictures:

r, fix z generic point near r (or r).



Note $T_z(B) \ni \vec{r0}$ displacement (makes sense in linear mfd.)

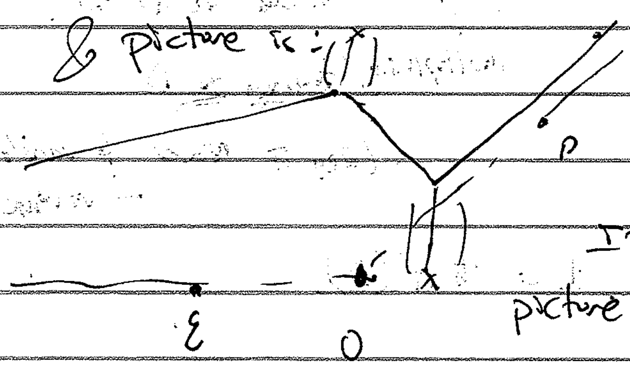
From q to z

Balancing of tangent vectors at 

Take case Θ_0.

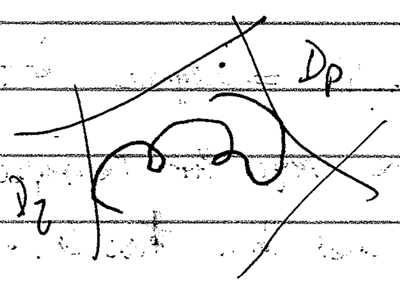
Coeff$_{\Theta_0}(\Theta_p \cdot \Theta_q)$. Then $\vec{r} = 0$, & vector is gone.

& picture is:



honest tropical tree.

It's tropicalization of picture:



Picture for two-point function: Coeff$_{\Theta_0}(\Theta_p \cdot \Theta_q)$.

Want mult. rule on $V = \bigoplus_{q \in B(\mathbb{Z})} \cdot \vartheta_q$.

$\operatorname{Spec}(V) = (Y$ w/ max'l boundary w/ max Ω, mirror

(in hol.-symp. case: deformation of original U).

Main problem: ∞-many ϑ_r can contribute to $\vartheta_p \cdot \vartheta_q$

(Necessary + sufficient) condition to imply finiteness:

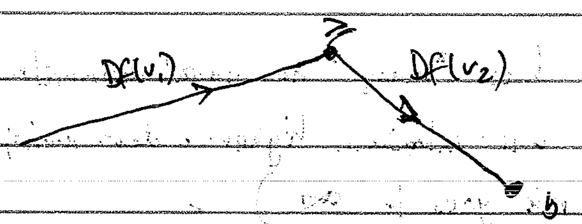
Broken lines give a notion of convexity on B.

Def: $f: B \to \mathbb{R}$ piecewise linear.

Call it "min-convex" (morally, min. of linear functions)

if D_f is decreasing (non-increasing) along all broken lines.

e.g.



Note: $\mathbb{R}^n$, & straight lines.

Then, min convex = min. of ~~convex~~ linear functions (i.e. "concave")

$\Rightarrow f: B \to \mathbb{R}$ min. convex

$\cong$ ↗ this is standard min-convex.

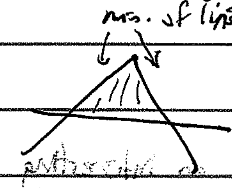
$\mathbb{R}^N = N_{S,\mathbb{R}}$

Now, suppose f min-convex, & have a picture, so

ϑ_r appears in $\vartheta_p \cdot \vartheta_q$.

Then, can check: $f(r) \geq f(p) + f(q)$.

$\Rightarrow$ if $Q_f := \{q \mid f(q) \geq -1\}$ "convex polygon" in B.

like 

min-convex = "concave"

Note: If Q_f is bounded, ~~this is~~ then

$\{r \mid f(r) \geq f(p) + f(q)\}$ is a finite set.

Then, mult. rule is polynomial (don't need Novikov ring).

[K, T-Mandel]

Conj (Thm in dim. 2): If $\varphi : U \longrightarrow \mathbb{A}^1$ regular function, then

$$\varphi^t : B \longrightarrow \mathbb{R} \text{ is min convex.}$$

Note: Say $f_1, \dots, f_r$ min convex, then $f = \min(f_1, \dots, f_r)$ min convex.

Suppose $\varphi_1, \dots, \varphi_r : U \longrightarrow \mathbb{A}^1$; let $f_i = \varphi_i^t$. $U^t(\mathbb{Z})$

Then, Q_f is bounded $\Longleftrightarrow$ for any ∂ divisor $q \in B(\mathbb{Z})$,

at least one φ_i has a pole at q.

$\varphi^t(q) = \mathrm{val}_{D_q}(\varphi)$, so it's negative $\Leftrightarrow$ pole at divisor.

If U is affine and $\varphi_1, \dots, \varphi_m$ gen. $\Gamma(U, \mathcal{O}_U)$, then this holds.

dim. 2 $\Longleftarrow$ (characterizes affines).

Note: any min convex f puts a filtration on $\bar{V}$.

Let $\tilde{V} \subset \bar{V}[t]$ subspace gen. by

$\vartheta_q \cdot T^s$, where $f(q) \geq -s$. (polygon in $B(\mathbb{Z})$)

$\tilde{V}$ is a graded subalgebra.

$$\mathrm{Spec}(\bar{V}) \subset \mathrm{Proj}(\tilde{V})$$

$\shortparallel$ $\quad\uparrow$ If Q_f is bounded, this is projective

$U^{\vee}$

Conj: This is a minimal model, and all minimal models come this way.

Thm: If f with Q_f bounded exists in the cluster case:

$\Rightarrow$ $\mathrm{Spec}(\bar{V})$ is affine CY w/ max'l boundary

$\mathrm{Proj}(\tilde{V})$ is a minimal model.

each choice of seed s identifies

$$Q_f \subset B(\mathbb{R}) = N_{\mathbb{R},s} = \mathbb{R}^N.$$

We get a canonical degeneration of $\mathrm{Proj}(\tilde{V})$ to $TV(Q_f)$

(e.g. Grassmannians $\rightsquigarrow$ get millions of degenerations of Grassmannians to toric varieties).

Q: What's the superpot'l on U corresponding to $\mathrm{Proj}(\tilde{V})$?

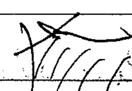
~~$A^t(\mathbb{Z}) \leftrightarrow$ basis of functions~~ $\quad A^{t} \longrightarrow$ basis of fns on $\mathcal{X}^t(\mathbb{Z})$

Take φ regular function on A. $\quad$ get subset of $\mathcal{X}^{trop}(\mathbb{Z})$ (boundary divisors).

$\varphi^{trop}: A^t \longrightarrow \mathbb{R}$ min. convex, $\qquad \mathcal{X}$

$Q_\varphi: \varphi^t \geq -1 \rightsquigarrow \mathrm{Proj}(\tilde{V}_\varphi) \supset \mathrm{Spec}(V)$

minimal model $\qquad\qquad$ basis of functions on A.



$x_i \in \mathcal{X}^t(\mathbb{Z})$.

Conj: Θ functions that appear in $\varphi = c_1\vartheta_{x_1} + \cdots + c_s\vartheta_{x_s}$.

<u>Also</u>:

$A^+ \times \mathcal{X}^+$ $\qquad\qquad$ θ fcn. corresponding to Σ_Y

$$\begin{tikzcd}
(a, x) \arrow[r] \arrow[d] & \mathrm{val}_{D_x}(\theta_a) \\
\mathrm{val}_{D_a}(\theta_x) \arrow[ur, phantom, "="] &
\end{tikzcd}$$

dim. 2, w/ T. Mandel

Keel III:

$$\mathcal{A} = \bigcup_{s\in\Gamma} T_{N,s} \quad \text{: union of tori glued by}$$

N lattice. $\qquad m_{s,s'} : T_{N,s} \dashrightarrow T_{N,s'}$

$$\mathcal{X} = \bigcup_{s\in\Gamma} T_{M,s} \qquad \begin{matrix} \| \\ m_{(v,F)} \\ \updownarrow \text{switch} \\ m_{(F,v)} \end{matrix} \qquad v \in N \qquad F \in M = N^* \quad F(v)=0$$

• N, $\langle\, ,\, \rangle$ skew symmetric given.

$$\overset{v}{\underset{s}{\|}} \in N$$

all mutations $F = \langle s, - \rangle$, of the form $m_{(s, \langle s, - \rangle)}$

Seed: ordered basis

$$S = (s_1, \dots, s_N) \text{ of } N, \quad s_k \in S \quad m_{s_k}.$$

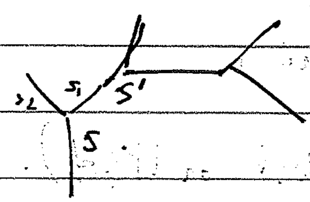
$$S' = (s_1', s_2', \dots, s_N')$$

$$s_i' = m_{s_k}(s_i) \quad i \neq k.$$

$$s_k' = -s_k = -m_{s_k}(s_k)$$

N-regular tree

e.g. $N=3$,



cluster alg. label edge by seed, for path, glue by mutation

Fock–Goncharov: $\mathcal{A}$ and $\mathcal{X}$ are mirror:

$$\mathcal{A}^+(\mathbb{Z}) \xrightarrow{\text{basis}} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

and vice versa.

Remark: FG conjecture wildly false: "most of the time" $\mathfrak{X}$ has no functions, whereas $\mathbb{A}^t \overset{set}{\cong} N$, infinite.

Conj: FG holds $\longleftrightarrow$ $\mathfrak{X}$ is affine up to flops, i.e.,

$\Gamma(\mathfrak{X}, \mathcal{O}_X)$ fin. gen., and

$\mathfrak{X} \longrightarrow \operatorname{Spec}(\Gamma(\mathfrak{X}, \mathcal{O}_X))$ iso outside codim. 2

FG's:

Double construction

let $L = N \oplus M$, $[(n_1, m_1), (n_2, m_2)] = \langle n_1, n_2 \rangle + m_2(n_1) - m_1(n_2)$

matrix: $\left(\begin{array}{c|c} B & +I \\ \hline -I & 0 \end{array}\right)$ $\quad U = N \oplus L$.

Now, can take $\bigcup_{s \in F} T_{L,s}$, b/c $m_{s,N} \subset L$.

$$\overset{\shortparallel}{D} \text{ double.}$$

Can also do dual:

$\mathcal{E} = \bigcup_{s \in I} T_{L^*, s}$. and unimodularity $\Rightarrow \mathcal{D} \cong \mathcal{E}$.

There are two maps

$$\begin{array}{ccl} L & \longrightarrow M & (n,m) \longmapsto m \\ \downarrow & & (n,m) \longmapsto \langle n, - \rangle + m. \\ M. & & \end{array}$$

This gives quotient by T_N.

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{L} \\ \mathcal{X} & \end{tikzcd} \quad ; \quad T_N \curvearrowright \mathcal{D}$$

other map $\mathcal{B}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, phantom, "\subset"] & \mathcal{D} \arrow[d] \\ & \mathcal{X} \end{tikzcd}$$

Upshot: FG for

$\mathcal{D} \Longleftrightarrow \mathcal{E}$ $\mathcal{B}$ $\Longleftrightarrow$ FG for $\mathcal{X}$ and $\mathcal{A}$

Note: $S = (s_1, \ldots, s_N)$; so $T_{M,S}$ has coordinates $z^{e_i} = X_{i,S}$. Then, at least

$T_{L^*,S}$ has $z^{e_i} = X_{1,S}$.

There's a canonical (partial) compactification due to FG "special completion":

$$\mathcal{X} \subset \hat{\mathcal{X}}$$

$$\bigcup T_{M,S} \subset \bigcup \mathbb{A}^N_{(X_{i,S})} \ni O_S.$$

Thm: $\mathcal{D}$ extends to $\hat{\mathcal{D}}$ and

$$\begin{tikzcd} \mathcal{D} \arrow[d] & & \hat{\mathcal{D}} \arrow[d] \\ \mathcal{X} & & \hat{\mathcal{X}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, phantom, "\subset"] \arrow[d] & \hat{\mathcal{D}} \arrow[d] & T_{N,S} \arrow[l, "\text{torus}"'] \arrow[d] \\ \mathcal{X} \arrow[r, phantom, "\subset"] & \hat{\mathcal{X}} & O_S \arrow[l, phantom, "\supset"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{D} \arrow[d] & \supset & \bar{\mathbb{D}} \arrow[d] \\ \hat{\mathcal{X}} & \supset & \bar{\mathcal{X}} \end{tikzcd}$$

formal completion along 1-strata

Thm: FG conjecture is true here in formal nbhood close to limit.

$$\mathcal{O}_{\bar{\mathbb{D}}} = \bigoplus_{q\in X^{trop}} R\,\mathcal{O}_q \qquad R = \mathcal{O}_{\bar{\mathcal{X}}}.$$

Goal: Find conditions ~~[illegible]~~ so these formal functions extend.

$$\Xi \qquad V = \bigoplus_{q\in\Xi^+(\mathbb{Z})} k\cdot\mathcal{O}_q.$$

(1) • Broken lines make sense, hold

(2) • V is commutative, associative, f.g.

(3) • Spec(V) is CY.

(4) • Spec(V) is the mirror

it's the FG mirror.

Conjecture: FG holds (1-4) iff
∃ bounded polygon in $\Xi^+(\mathbb{R})$
(i.e. min-convex fn, that gives convex set.)

Thm: If ∃ bounded polygon, then (1-3) hold
& $\Gamma(\mathbb{D},\mathcal{O}_{\mathbb{D}}) \subset V$.

(upper) cluster algebra (w/ geometric type).
(get same versions for $\mathcal{A}$ and $\mathcal{X}$).

want a canonical basis on something we know.

Thm: (1-4) if ∃ seed with a "maximal green sequence."
⇒ if ∃ an acyclic seed.

tells you intersection #'s of basis.

$\langle\,,\rangle$ + Basis ⟺ quiver.

No oriented loops = acyclic (e.g. Dynkin diagrams).

Thm: FG holds for all cluster varieties for
$Hom(\pi_1(X), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$ X punctured Riemann surface

(already exists, but this basis was constructed using Riemann surface
⇒ then doesn't ~~so~~ use ~~a~~ Riemann surface to give a canonical basis!!!).

only prove bases are the same in 1-case (Nietzke)
$\mathbb{P}^1$ w/ 4-punctures.

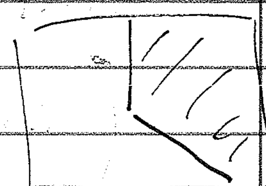
- K.S. Scattering Diagram:

W lattice $(L_S^\#)$

Scattering diagrams: bunch of walls

Walls! • Γ = full dim. convex cone. $\subseteq F^\perp$, $F \in W^*$

also, a vector $\vec{v} \in \Gamma$, where
$\vec{\Gamma} = \vec{\Gamma} + \mathbb{R}_{\geq 0} \cdot \vec{v}$.



$F^\perp \subseteq \mathbb{R}^3$

convex cone
(whole space,
half-space,
etc.)

a scattering function
$$f_\Gamma = \sum_{n \geq 0} a_n z^{-n \cdot v} \in \mathbb{Q}[Z_{\leq 0} \cdot v].$$
(think as finite, though can be formal)

(determines $\vec{v}$).

(Don't need Artinian rings in this context!)

Wall gives automorphism
Aut of T_{W^*}
monomial in
$$z^w \longmapsto z^w (f_\Gamma)^{F(w)}$$

called consistent, if composition around any loop is id.

(Gross-Siebert)

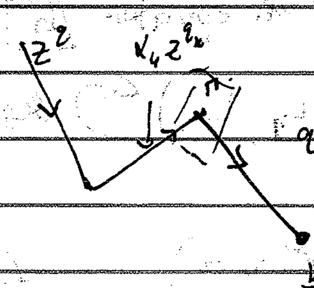
K S lemma: ~~there's~~ Given any diagram, there's a canonical way to add more walls, so it's consistent.

In L_S^* : initial walls; $s_i^\perp$, scattering function

$S = (s_1, \dots, s_N) \in N \subset L$ $\qquad 1 + z^{\langle s_i, - \rangle} \overset{GS/KS}{\rightsquigarrow} F_S$ scattering diagram

~~Broken~~ Now, a broken line in W:

is a ~~g~~ jagged path with



$q_{k+1} = q_k - n^\vee$.

from q to b:

- $-q_k$ parallel to edge.
- bends occur at a wall.
- $\alpha_{k+1} z^{q_{k+1}} = \alpha_k z^{q_k} \cdot (f_p)^{F(-q_k)}$

defining function of wall.

Some monomial from ~~this~~ (get to choose how much to bend)

Cluster complex: $\qquad s = (s_1, \dots, s_N)$

Each $T_{L^*, s}$ comes with $X_{i,s}$

$\downarrow$ coordinates

$T_{M,s}$ — $\qquad x_{i,s} = X_{i,s}^*$ $\;\Vert\;$ s_i

$C_s \subset \mathcal{E}^+(\mathbb{R}) = \{ q \mid x_{i,s}(q) \geq 0 \; \forall i \}$

chamber

$\mathcal{E}^+$

Thm: (Kontsevich) The cluster complex is part of the support of the scattering diagram, and in a chamber, there are no other walls.

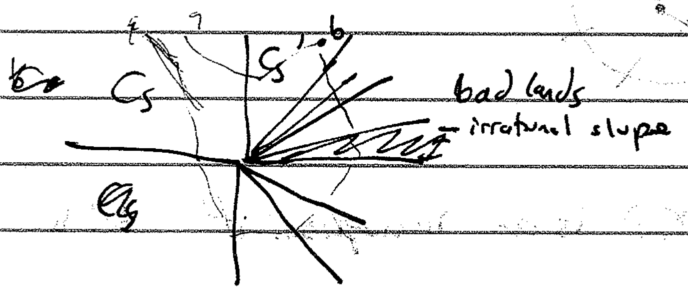
$\downarrow$

$\mathcal{X}^+ \supset C_s$ in seed

" $M_{\mathbb{R}}$ " "First quadrant"

$x_{i,s} = s_i \geq 0$: ☒

In $\mathcal{X}^{trop}(\mathbb{R})$ in dimension 2:



In L_S^*

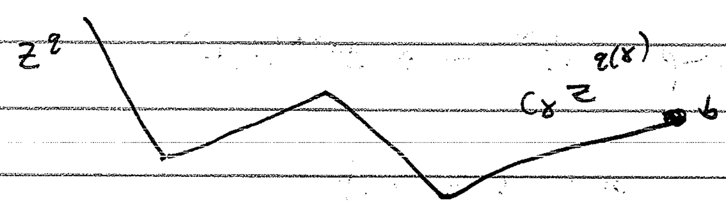
$$\mathcal{E} = \bigcup_S T_{L,S}, \text{ \& wall crossing: } \longrightarrow \text{Aut}^{\text{rational}} / \text{mutation of } T_L$$

$= \text{gluing mutation.}$

choose $b \in C_{S'}$, choose $q \in \mathcal{E}^+$

Define $\Theta_{q,b} = \sum_{q \to b} c_\gamma z^{q(\gamma)}$

(CPS)



Main result:

$\Theta_{q,b}$, $\Theta_{q,b'}$ $\in \Gamma(T_L)$, $\Gamma(T_{L'})$

~~are related by mutation.~~

~~(this could be a formal power series).~~

~~(move b in chamber: get same fcn.)~~

$\cup T_{L,S}\, c_\gamma z^{q(\gamma)}$

So, $\Theta_{q,b} \in \Gamma(\mathcal{E}, \mathcal{O})$

We get FG or nose if:

only finitely many $\xrightarrow[\gamma]{} b$

If not: $\rightsquigarrow$ Formal FG.) $\Rightarrow$ mult. rule on V is associative

(because its functions on $\overline{D}$, avoids any A_∞ stuff).

To get FG: need condition to ensure

$\# \gamma \xrightarrow{q} b < \infty$.

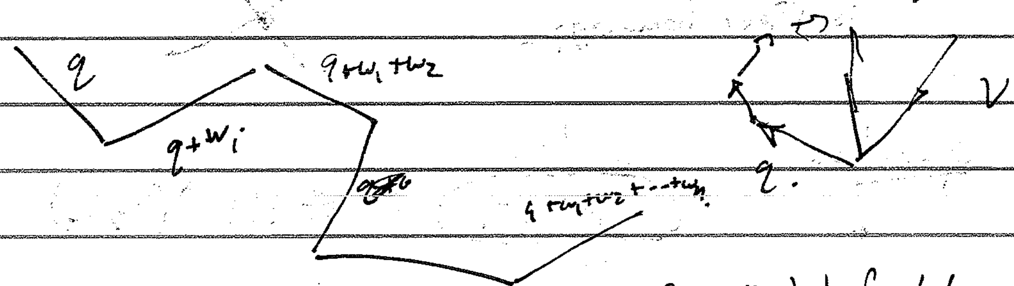
Thm: $\# < \infty$ if $\exists$ acyclic seed.

Pf: Note initial diagram had scattering monomials

living in L^*_s. $z^{[s_i, 0]}$ $[\ ,\ ]$ unimodular in L.

I_s = cone in L^* generated by $[s_i, 0]$ $i = 1, \dots, N$

s_i basis of N. is strictly convex in lattice I_s.

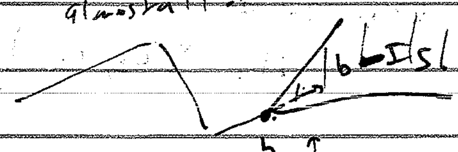


where $w \in I_s \setminus \{0\}$

for all but finitely many γ $q \to b$,

The final monomial $z(\gamma) \in I_s$.

for almost all:



when bend, direction pointing in (neg. of) convex cone at b.

Now use the fact:

F_s is independent of the seed (not strictly true, but nearly ...).

$L^*_{s,\mathbb{R}}$, $\xrightarrow{m^t_{s,s'}}$ $L^*_{s',\mathbb{R}}$

F_s $\quad F_{s'}$ $= m_{s_e}(s)$,

$m_{s',s}(F_{s'}) = F_s$, with one change (need to reverse sign of one vector).

and broken lines are invariant. (broken $\Leftrightarrow$ broken).

Apply here: take any string of mutations

$s \rightsquigarrow s'$

Then same thing holds: for all but finitely many, final monomial $q(\gamma)$ lands in corresponding cone, but in other seed.

i.e. the cone $\left(D_b\, m_{s,s'}\right)^{-1}\left(I_{s'}\right)$

~~Step~~ Rmk: All mutations are PL w.r.t. cluster complex.

What if $D_b\, m_{s,s'} = Id$?

Then, for all but finitely many γ,

$$q(\gamma) \in I_s \cap I_{s'}$$

$\quad ||$

Span $[s_i, -]$ $\quad [s_i', 0)$

s acyclic $\Longleftrightarrow$

$(s_1, \dots, s_N)$ (after reordering),

$\langle s_i, s_j \rangle \geq 0$ for $j \geq i$.

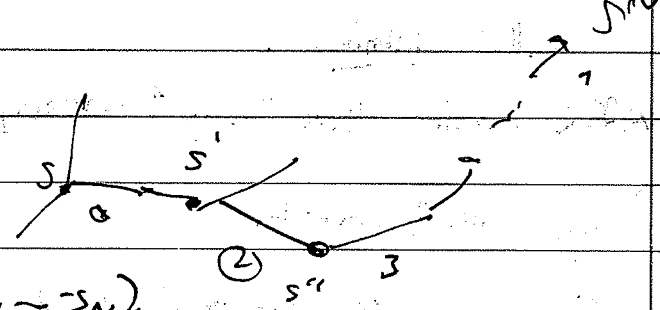
$\Rightarrow$ ~~mutate~~ if mutate:

$D_b . m_{s,s'} = Id$

$b \in C_s$

and $s^{(N)} = -s$

$= (-s_1, -s_2, \dots, -s_N)$.



and certainly, $I_s \cap I_{-s} = \{0\}$, so there are no such lines!

Something interesting:

$$N \longrightarrow M$$
$$v \longmapsto \langle v, - \rangle$$

get exact sequence:

$$0 \to K \to N \to M \to K^*$$
$$K = \ker(\text{map})$$

$\Longrightarrow$

$T_K \curvearrowright A$ $\qquad$ $\pi^{-1}(e) = U \in \mathfrak{X}$

$A \xrightarrow{/T_K} U$ $\qquad$ $\downarrow \quad \downarrow$

$\qquad\qquad e \in T_K^*$

$\mathfrak{X}^{\text{trop}}$ $\quad$ and $T_K \curvearrowright A$, so $\quad H^0(U, L)$

$\downarrow$ $\qquad \Gamma(A, \mathcal{O}) = \bigoplus_{L \in K^*} \Gamma(A, \mathcal{O})^L$

$L \in K^*$

$= \operatorname{Pic}(U)$

$\pi^{-1}(L)$ basis of this weight space.

Prop: $\operatorname{Pic}(U) = K^*$, and

$$A \longrightarrow U$$

$\uparrow$ is the universal torsor (canonical torus bdle.)

Fibre $T_{\operatorname{Pic}(U)^*}$

$\Longrightarrow$

$$\Gamma(A, \mathcal{O}_A) = \operatorname{Cox}(U) = \bigoplus_{L \in \operatorname{Pic}(U)} H^0(U, L)$$

$\pi^{-1}(L) \subset \mathcal{X}^t(\mathbb{Z})$

basis for $H^0(U, L)$ says that all line bundles have "same #" of sections.

only happen if U is affine.

Choosing a seed $\mathcal{X}^{trop}$ $\bar{N}$ "N/k"

$$\begin{array}{c} \mathcal{X}^{trop} \\ \downarrow \pi \\ \mathcal{X}^{*} \end{array}$$

Wouldn't it be great if there was a canonical polygon.

Q: ~~Does $\exists$ some canonical polygon~~

Could it be that all ~~of~~ U (affine or not) have canonical basis for all $H^0(U, L)$?

- $\exists$ canonical polygon $Q \subset \mathcal{X}^{trop}$ such that $\pi^{-1}(L) \cap Q$ is a basis.

$$Q \subset \mathcal{X}^{trop} \xrightarrow{\pi} \mathrm{Pic}(U)$$

$\mathrm{Hom}(A, A^!)$

$A \otimes A^!$

Hope: ~~Q comes from tropicalization of a potential,~~

~~but $\mathcal{X}$ may not have fns, & currently ~ global conditions~~, so not quite.

$\mathrm{Hom}(A^!, A^*)$

$A \otimes B =$ ~~$\mathrm{Hom}(A \otimes A, A \otimes B)$~~

$A = A^{!!}$

A