Kontsevich I: Complex moduli of LG models

X Fano, $\dim_{\mathbb{C}} X = n = \dim_{\mathbb{C}} Y$ LG model

smooth proj $/\mathbb{C}$ $\xleftrightarrow{\text{MS.}}$ $\begin{array}{c} Y \\ {\scriptstyle w}\downarrow \\ \mathbb{A}^1 \end{array}$ smooth quasi-projective $K_Y = 0$.

- $-K_X$ very ample
- choose $s \in \Gamma(X, -K_X)$

$D_X = s^{-1}(0)$.

Crit(W) compact, but may not necessarily proper.

Versions: • D_X smooth $\longleftrightarrow$ W is proper.

• D_X normal crossings $\longleftrightarrow$ fibers of W = Zariski open in (n-1) CYs.

Ex: $X = \mathbb{CP}^n$ mirror $Y = (\mathbb{C}^*)^n$

D_X = toric.

$$z_1, \dots, z_n \mapsto \sum z_i + \frac{1}{\prod z_i} \qquad Y \xrightarrow{W} \mathbb{C}$$

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if D_X smooth deg = n+1 CY hypersurface get $Y' \supset Y$ some (fiberwise) compactification

intermediate smoothings $\longleftrightarrow$ partial compactifications

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$$\mathcal{F}(X) = \coprod_{\substack{\text{eigenvalues} \\ \text{of } K_X *}} \mathcal{F}_\lambda(X) \sim D^b \mathrm{Sing}(W) = \coprod_{\lambda \in \mathrm{Crit}\, W} D^b_{\mathrm{sing}}(Y, w-\lambda).$$

($*$ quantum)

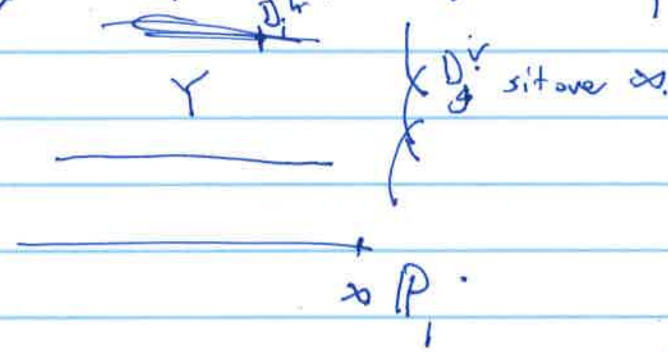
$$\mathcal{F}_{\text{wrapped}}(X - D_X) = D^b \mathrm{Coh}(Y).$$

$\mathrm{rk}\, H^\cdot(X \setminus D_X) = \mathrm{rk}\, H^\cdot(Y)$.

<u>Framework</u>: $\begin{array}{c} Y \\ \downarrow w \\ \mathbb{A}^1 \end{array}$ smooth, qproj., $K_Y = 0$, fix $\mathrm{vol}_Y \in \Gamma(Y, K_Y)$

Fix $Y \subset \overline{Y}$ smooth projective compactification.

Assume $D_Y = \overline{Y} \setminus Y$ simple normal crossing $= \bigcup_i D_i^h \cup \bigcup_j D_j^v$ (horizontal, vertical)



Assume $\mathrm{ord}_{D_i^h} W = 0$ and $\overset{(*)}{\mathrm{ord}}_{D_j^v} W = 1$.

and $W|_{\overline{D_i^h}}$ is a fibration over $\mathbb{P}^1$.

And $\mathrm{ord}_{D_i^h} \mathrm{vol}_Y \overset{0,-1}{=}$, $\mathrm{ord}_{D^v} \mathrm{vol} = -1$

these are <u>simplifying</u> <u>assumptions</u> : why $(*)$? (in principle, could be something else).

Take $\cdot\ H^*(Y, w^{-1}(re^{i\theta}))$

$r >> 1$.

want θ monodromy to be <u>unipotent</u>. (in general quasi-unipotent).

And $(*)$ guarantees unipotent

(mirror to $K_X *_{\text{quantum}} - \_\_$).

Motivation:

If this was mirror to X, symp., then symplectic parameters should correspond to complex parameters. In particular should get a <u>smooth</u> <u>moduli</u> space

Claim: Deformation theory of

$$(\bar{Y}, (D_i^h), (D_j^v), w, \mathrm{vol}_Y)$$

is unobstructed.

Technical condition: $H^1(\bar{Y}, \mathbb{Q}) = 0$ (hard to imagine why not true: rational variety).

Corollary: Consider $\bar{Y}$. smooth projective $H^1(\bar{Y}, \mathbb{Q}) = 0$

$$\begin{array}{c} \bar{Y} \\ w \downarrow \\ \mathbb{P}^1 \end{array} \qquad c(K_{\bar{Y}}) = -w^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1))).$$

Then, deformation theory of is unobstructed.

Get a map $\mathbb{P}^1 \longrightarrow$ moduli of $(n-1)$ CY.

(smooth by Tian-Todorov?)

Proof: forget about vol_Y. (can b/c $H^1(\bar{Y}, \mathbb{Q}) = 0$. vol still exists for small deformation of divisors ...).

Deformation theory: sheaf of L_∞-algebras

$$\mathfrak{g} \qquad \mathfrak{g}^0 \xrightarrow[m_1]{} \mathfrak{g}^1 \quad + \text{higher} \ldots$$

$\mathfrak{g}^0$ = vector fields on $\bar{Y}$ tangent to all D_i^h, D_j^v.

and $\mathfrak{g}^1 = w^*(\text{v. fields on } \mathbb{CP}^1 \text{ vanishing at } \infty)$.

(as Coh sheaf) vector fields preserving point.

$$T_{\bar{Y}, D_{\bar{Y}}} \longrightarrow w^*(T_{\mathbb{P}^1, \infty})$$

Main Def: $\tilde{\Omega}^0$ = coh. sheaf on $\bar{Y}$ ~~...~~ on log-forms α relative to $D_{\bar{Y}}$.

Recall: $D_{\bar{Y}}$ n.c. divisor $z_1 \cdots z_k = 0$ in $\mathbb{C}^n$, $n \geq k$.

gen. by $\wedge\left(\frac{dz_i}{z_i}_{i \leq k}, dz_{j}{}_{j > k}\right)$.

such that $dW \wedge \alpha$ is also a log form

One can check: it is a vector bundle.

Two differentials:

$$d, \quad dW \wedge \cdot \quad \tilde{\Omega}^i \longrightarrow \tilde{\Omega}^{i+1}.$$

Translate to polyvectorfields with $\lrcorner\, \mathrm{vol}_Y$: Get class $\tilde{\mathfrak{g}}$.

Get:

Fact: the image is closed under $[\ ,\ ]$.

Ex: One variable: t local coordinate

$|t| << 1$, $w = t^{-1}$.

$$\mathrm{vol} = \frac{dt}{t} \quad (\text{remove } 0).$$

log forms: deg:

$$\begin{array}{cc} 0 & 1 \\ \hline 1 & \frac{dt}{dt}. \end{array} \qquad dw = \frac{-dt}{t^2}.$$

$\tilde{\Omega}$: get generators $\cdot t$ and $\frac{dt}{t}$.

as vector fields: get $t^2 \frac{\partial}{\partial t}$, 1.

$$\tilde{\mathfrak{g}}^{1-n} \longrightarrow \cdots \longrightarrow \tilde{\mathfrak{g}}^0 \longrightarrow \tilde{\mathfrak{g}}^1.$$

sheaf of dg Lie algebras (diff. $= [W, \cdot]$.

sub sheaf: last 2 terms $\tilde{\mathfrak{g}}^0 \longrightarrow \tilde{\mathfrak{g}}^1$: $\quad (\tilde{\mathfrak{g}}^i = \tilde{\Omega}^{n+1-i})$.

Compare to our

$$\mathfrak{g}^0 \longrightarrow \mathfrak{g}^1 \quad L_\infty \text{ algebra defined before}$$

Can immediately see

$$(\widetilde{\mathfrak{g}}^0 \to \widetilde{\mathfrak{g}}^1) \to (\mathfrak{g}^0 \to \mathfrak{g}^1)$$

(map to open part)

In fact: get an L_∞ quasi-isomorphism of sheaves, so can replace

$$(\mathfrak{g}_0 \to \mathfrak{g}^1) \quad \text{by} \quad (\widetilde{\mathfrak{g}}^0 \to \widetilde{\mathfrak{g}}^1). \quad \text{dg Lie replacement.}$$

(much nicer to work with!).

Want: $R\Gamma(\bar{Y}, \mathfrak{g}^0 \to \mathfrak{g}^1)$ is homotopy abelian

$\iff R\Gamma(\bar{Y}, \widetilde{\mathfrak{g}}^0 \to \widetilde{\mathfrak{g}}^1)$ is homotopy abelian

(i.e. quasi-iso to cohomology??)

Actual proofs:

(1) $R\Gamma(\bar{Y}, \underset{\text{big}}{\widetilde{\mathfrak{g}}}, [w, \cdot])$ is homotopy abelian

(2) Truncation $R\Gamma(\bar{Y}, \widetilde{\mathfrak{g}}^0 \to \widetilde{\mathfrak{g}}^1) \longrightarrow R\Gamma(\bar{Y}, \widetilde{\mathfrak{g}}, [w, -])$

is $\iff$ on cohomology. (as in Tian–Todorov -- hard part). divergence? dw?

To prove (1): Consider $R\Gamma(\bar{Y}, \widetilde{\mathfrak{g}}[[\hbar]], [w, \cdot] + \hbar\, \mathrm{div})$.

enlargement, deformed differential. map $\gamma \longrightarrow e^{\gamma/\hbar} - 1$ can write explicit L_∞ map proving homotopy abelian.

in limit $\hbar \to 0$.!

Double degeneration of spectral sequences

Claim: $\mathrm{rk}\ R\Gamma(\bar{Y}, \tilde{g}, c_1 \operatorname{div} + c_2[w, \cdot])$

is indep. of $(c_1, c_2) \in \mathbb{C}^2$. (vector bdle over $\mathbb{C}^2$)

To prove (2): If ~~...~~ $c_2 = 0$, then obviously including 6

~~...~~ ~~...~~ cohomology rank doesn't jump.

Translate back to forms on $\tilde{\Omega}$:

$\mathrm{rk}\ R\Gamma(\bar{Y}, \tilde{\Omega}, c_1 d + c_2(dw \wedge -))$

is independent on $(c_1, c_2) \in \mathbb{C}^2$.

(get vec. bdle over $\mathbb{C}^2$).

Prop: This holds $\forall$ $w: Y \to \mathbb{A}^1$ extending to $\bar{Y}$

forget about volume form

~~...~~ $\mathrm{ord}_w D^v = -1$

$\mathrm{ord}_w D^h = 1$

Lemma 1: ~~$\mathrm{rk}\ R\Gamma(...)$~~ $\mathrm{rk}\ R\Gamma(\bar{Y}, \tilde{\Omega}, 0 \text{ differential})$

$= \mathrm{rk}\ R\Gamma(\bar{Y}, \tilde{\Omega}, d)$ De Rham.

Proof: imitating Deligne–Illusie methods

use reduction mod $p > Y$

$\tilde{\Omega}$, 0 differential $\to \Omega^\cdot$ de Rham pull back by Frobenius

Locally: $f_0, df_1, \dots, df_k \longrightarrow f_0^p f_1^{p-1} \cdots f_k^{p-1} df_1 \cdots df_k$

quasi-iso,

(get quasi-iso of all forms w/ closed forms).

Case: 1 variable

$\tilde{\Omega}$ $\quad$ t $\quad$ $\mathbb{F}_p[t]$

	0	1
	t	$\frac{dt}{t}$

Consider basis of 1-forms:

$$\begin{array}{ccc} t & \xrightarrow{d} & dt/t \\ t^2 & & t\,dt/t \\ \vdots & & \\ t^p & & t^p \frac{dt}{t} \\ t^{p+1} & & \vdots \\ \vdots & & \\ t^{2p} & & t^{2p}\frac{dt}{t} \\ \vdots & & \end{array}$$

circled: survive to cohomology. local calculation. (Saito: alternate proof using pure Hodge theory)

Lemma ② $\operatorname{Rk} R\Gamma(\tilde{\Omega}, d) = \operatorname{rk} H^\cdot(Y, w^{-1}(-\infty))$.

$(\tilde{\Omega}^\cdot, d)$ $\sim$ in analytic topology to $D^b_{constr.}(\bar{Y})$ constructible.

what is it?

take Y, change to nbhd w/ boundary, push forward something to glbl sup, constr sheaf

Lemma ③: $\operatorname{rk} H^\cdot(Y, w^{-1}(-\infty))$
$= \operatorname{rk} R\Gamma(Y_{Zar}, \Omega, d+dw)$ — Malgrange Fourier-transform of D-modules
$= \operatorname{rk} R\Gamma(Y_{Zar}, \Omega, dw)$ — Barannikov-Kontsevich-Sabbah degeneration

Lemma ④: $\operatorname{rk} R\Gamma(Y_{Zar}, \Omega, d+dw) = \operatorname{rk} R\Gamma(\tilde{Y}, \tilde{\Omega}, d+dw)$
similar to Grothendieck's deRham = Betti (passes through log forms).

Fukaya–Seidel

Conclusion: $\mathrm{rk}\, H^\cdot(Y, w^{-1}(-\infty)) = \mathrm{rk}\, HP^\cdot(FS(Y,w))$ (+)

$$= \bigoplus_{0 \le i,j \le n} \mathrm{rk}\, H^i(\overline{Y}, \tilde{\Omega}^j)$$

Q: (+) ?? Explanation:

Now Assume no horizontal divisors:

then duality $\tilde{\Omega}^{n-j} = (\tilde{\Omega}^j)^* \otimes \Omega^n_{\overline{Y}}$ (Poincaré duality)

Suggests: $\mathrm{rk}\, H^i(\overline{Y}, \tilde{\Omega}^j) = \mathrm{rk}\, H^i(X, \Omega_X^{n-j})$

mirror dual Fano if it exists

actually get bigraded thing from nothing

Can reconstruct Hodge numbers on Fano variety.

(Hodge #'s before came from action of unipotent monodromy. Not here apparently).

(joint w/ Katzarkov – Pantev)