Kontsevich II: ~~Pre-CY algebras~~

weak CY algebras (joint w/ Y. Vlassopoulos).

compact case | smooth case

$\dim A < \infty$ A_∞ alg.

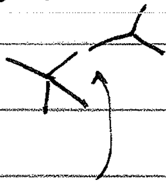
$A \in \mathrm{Perf}(\mathbb{C}\text{-mod})$

Def: CY structure $\Longleftrightarrow$ cyclic algebra structure

$\Longleftrightarrow$ solution of M-C equation in necklace like algebra

$(\ ,\ ) : A^{\otimes 2} \to \mathbb{C}$ non-deg., and

on cyclic invariants $\prod (A^{\otimes n})/\mathbb{Z}_n$ $[\ ,\ ]$ bracket

$= \sum$  using pairing.

$n \geq 3$ (poly-linear functionals, get A_∞ str. cyclic tensor product).

Classification: (M.K. + Y.S.)

Given A, need $\alpha : HC_*(A) \to \mathbb{C}$ of appropriate degree, such that

composition $HH_*(A,A) \xrightarrow{\text{natural}} HC_*(A) \xrightarrow{\alpha} \mathbb{C}$

gives a functional $\in (HH_*(A,A))^*$

$= R\mathrm{Hom}_{A\otimes A^{op}\text{-mod}}(A, A^*)$,

and this needs to give a quasi-isomorphism

$A \sim A^*$ (of certain degree, d).

e.g. deg $d \Rightarrow$

$A[d] \sim A^*$.

Good def'n

~~Predef.~~ [Ginzburg]

$A^\vee := R\mathrm{Hom}_{A\otimes A^{op}}(A, A\otimes A^{op})$

$A[-d] \sim A^\vee$ as bimodules

(not complete structure)

(w/o it, no classification & no calculus.)

Keller, vdB – wrong def'n!

Classification: (MK + YV) $\longleftrightarrow$ A A_∞ algebra

$\beta \in HC^-(A)$

$\searrow$ maps

$HH_*(A,A)$.

such that image of β is a quasi-iso,

considered as $R\mathrm{Hom}_{A\otimes A^{op}}(A^\vee, A) \cong HH_*(A,A)$

(recall $A^\vee := \mathrm{Hom}_{A\otimes A^{op}}(A, A\otimes A^{op})$)

compact case

A_∞ "stacks," CY algebras

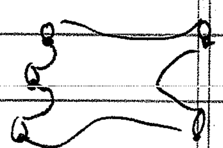
$\Longleftrightarrow$ f.d. A_∞ algs w/
this class in $(HC)^*$ satisfying
non-degeneracy.

PROP action for CY A.

$H = H_\cdot(A,A)$ Hochschild homology.

Then.

$H_\cdot(M_{g,\vec{n}_1,\vec{n}_2}) \otimes H^{\otimes n_1} \to H^{\otimes n_2}$



$n_1 \geq 1$,
$n_2 \geq 0$.

Same Prop, but

$n_1 \geq 0$

smooth.

$n_2 \geq 1$.

Some day, maybe we can compactify,
& remove arrows
(fixes tangent directions).

Def: Weak CY algebra A / char. 0 field.

no finiteness assumption at all.

A a $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ graded space.

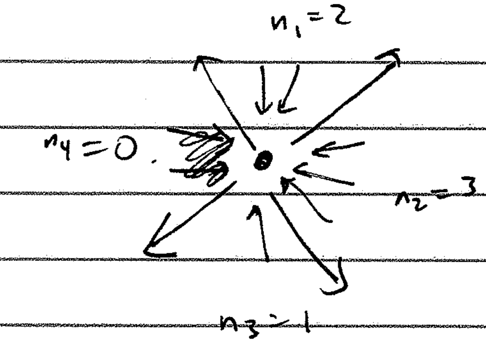
with operators $m_{n_1,\ldots,n_k} : A^{\otimes n_1} \otimes \cdots \otimes A^{\otimes n_k} \to A^{\otimes k}$

$\forall\ k \geq 1,\ n_1,\ldots,n_k \geq 0$, except $m_0 = 0$

$(m_\emptyset = 0$ too$)$.

these should be
cyclically invariant in this sense:

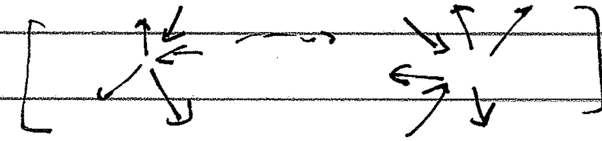
$k=4$



invariant under cyclic rotation.

And these shall satisfy Maurer–Cartan equation. w.r.t. bracket.

There's a bracket:



connect arrows going out to arrows going in

Seidel: same notion for f.d. algebras. Called boundary algebras
(responsible for $H^\cdot(M, \text{w/ boundary})$.)

$k = 1, 2, \ldots$

$k=1$ part is the same as A_∞ str.

commutator w/ $k=1$ part $[\mu, \cdot]$:= "Higher Hochschild complex"

$$C^{(k)}(A,A) := RHom_{A^{\otimes k} \otimes (A^{\otimes k})^{op} \text{ mod}}(A^{\otimes k}, \sigma_k A^{\otimes k}).$$

$\sigma_k := (1, \ldots, k)$ generator of cyclic permutation.
graph of cyclic permutation.

(A_∞ case, use bimodules, k-modules, etc.)

Then define

$$\prod_{k \geq 1} C^{(k)}/\mathbb{Z}/k\mathbb{Z}$$ is quasi-iso. to necklace Lie algebra

(not same as).

smooth case

$HH^{(k)}(A,A) \cong R\mathrm{Hom}_{\text{Endofunctor}}(Id, S^{1-k})$

↑ Serre functor ^ 1-k.

makes sense for smooth algebras, b/c

$S^{-1} := \otimes A^{\vee}$

(∵ S goes to infinite object)

compact case:

$= R\mathrm{Hom}_{\text{Endofunctors}}(S^{k-1}, Id)$

where $S = A^{*} \otimes_A \cdot$

k=2: first correction to A_∞ structure:

get elements here, and

ask them to be isomorphism in smooth & compact case.

Smooth variety: $\otimes K_X^{-1}$, cyclic group acts by mult of ± 1, ind. of dimension, triviality of cyclic group action is some derived commutativity

in general, for free algebras, σ_k acts very non-trivially.

<u>Examples of weak CY:</u>

1. Smooth scheme $X/\mathbb{C}$, + section s of K_X^{-1}.

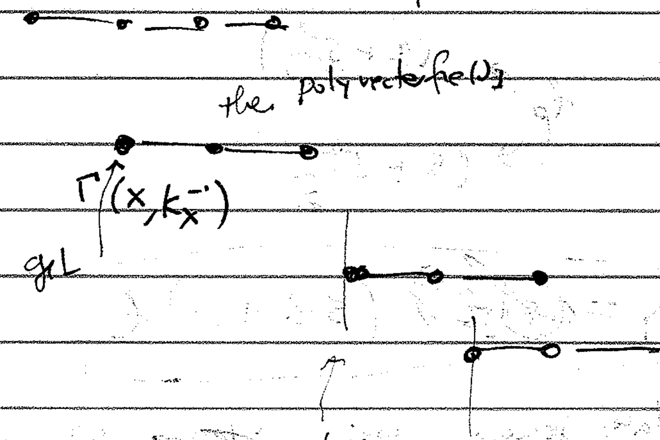
⇝ weak CY structure. s = first correction + A_∞ str.

why no higher corrections?

~~Deformation complex for higher HH:~~

$C^{(1)}$ 0 1 2 3 4

$C^{(2)}$ the polyvector fields



Point:
$s \in HH^{(2)}(A, A)$
$A = End(\text{any generator})$

If $\mathbb{Z}$-graded, simple

HKR - type: $HH^{(k)} := R\mathrm{Hom}_{X \times X}(\mathcal{O}_\Delta, K_{X,\Delta}^{\otimes 1-k})$, & cyclic group

action not obvious in this definition

Don't know that spectral sequence really splits.

So, polyvector fields w/ coeffs in

(??) Mirror dual:

$FS\begin{pmatrix} Y \\ \downarrow W \\ \mathbb{C} \end{pmatrix}$ should be weak CY category

2. M finite CW complex, connected m_0 basepoint.

$\mathcal{A} := \text{chains}(\Omega(M, m_0))$ top. monoid

if finite cell complex, neat model, &

$\mathcal{A}$ homologically smooth

$\forall \ \beta \in H_\bullet(M) \longrightarrow$ ~~weak~~ associate CY str.

If (M, β) Poincaré duality space (i.e. $\beta = [M]$)

$\rightarrow$ CY structure.

More generally, if M is singular Lagrangian in X
(small nbhd Weinstein contracts to L)

Expect: $HW^*(M, M)$ Homologically smooth, cpct if M cpct.

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3. ~~Seidel~~ (P. Seidel).

$(M, \partial M \neq \emptyset)$.

$C^\bullet_{\bar{q}}(M)$ is weak CY.

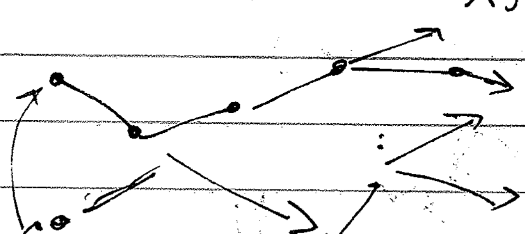
Graphical calculus:

$$H_\bullet(\mathcal{M}_{g, \vec{n}_1, \vec{n}_2}) \otimes H^{\otimes n_1} \longrightarrow H^{\otimes n_2}$$

$n_1, n_2 \geq 1$.

How? use

Acyclic oriented ribbon graphs. (no directed cycles)



3 types of vertices:

n_1 $\bullet\!\!\rightarrow$ $\deg_{in} = 0$

$\deg_{out} = 1$.

put (Hochschild complexes)

& suppose allways to apply operation

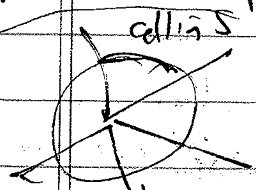
Simplest $A^0 \to A^{\otimes 2}$ inner vertices

$A^{\otimes 2} \to A^{\otimes 1}$

$\deg_{out} \geq 1$

if $\deg_{out} = 1$, then $\deg_{in} \geq 2$ (no differential)

cell in S'



Choose vertex + sector (complicated story w/ unit).

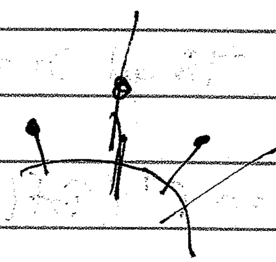
resulting $\deg_{out} \geq 1$ $\deg_{in} = 0$.

oriented graphs, not quite all ribbon graphs

Claims: gives new decomp. of moduli space of curves.

non oriented metrized ribbon graphs w/ n_1 ∞ legs
& n_2 marked vertices

all other vertices have deg ≥ 3



with metrics : get model for moduli of curves

functions $\Gamma \longrightarrow \mathbb{R}_{\geq 0}$ (graph)

distance to set of marked vertices.

(have some local maxima)

gradient flow gives an oriented acyclic graph.

Application: complete algebraization of
"string topology"

probably hard to relate geometric construction w/ K-V construction.

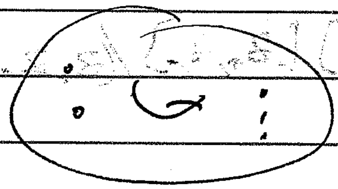
(Not terribly interesting example:)

For LG model $\begin{array}{c} Y \\ \downarrow W \\ \mathbb{C} \end{array}$ assume W proper

$HH_* := H^\cdot(Y, W^{-1}(-\infty))$

dual space $H^* = H^\cdot(Y, W^{-1}(\infty))$ ← different fiber.

GW invariants:

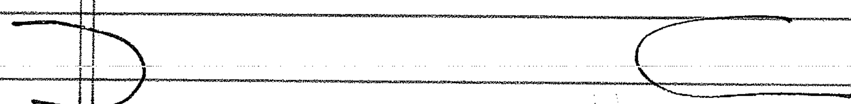


n_1 n_2

maps to vertical fibers.

(way to arrange GW invts. of vertical fibers in family slightly degenerate.)

n_1 lies on n_2 lies on



means, map to point

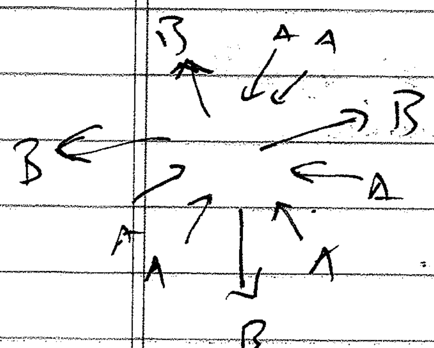
Weak CY algebras <u>form a category</u>

(what's a non-invertible notion of morphism? we know ....)

Morphism $A \longrightarrow B$

is a hom $A^{\otimes n_1} \cdots \otimes A^{\otimes n_1} \longrightarrow B^{\otimes k}$

$n_i \geq 1 \quad k \geq 1$



complicated rules for composing, differential.

<u>Origin</u>: weak CY algebras

non comm. generalization of $p \in L \subset X$ (base pt p) lagr. manifold in

X symply morally X := formal nhood of L.

(expand Homological vector field at p — replace A_∞ by A_∞ ——— ?)

Imagine $\dim A < \infty$

look at $A \oplus A^*[\cdot d]$, A_∞ str. on this s.t.

A is a subalgebra.

Micromorphisms (A. Weinstein + collaborators)

A map $L_1 X_1 \dashrightarrow L_2 X_2$

Morphism: $(X_1^{\circ}, -\omega) \times (X_2, \omega) \supseteq L_f$ lagr. submanifold.

such that $L_f \cap (L_1 \times L_2)$ is graph of a map

$\tilde{L}_f : L_1 \to L_2$ ← schematically, so clean intersection

(actual maps on lag's, but in sympl., you wiggle).

Miraculously, such things can be composed → no transversality issues.

Translate to language, get this notion of a weak CY-category.

Most difficult: Proof of classification of smooth CY algebras.

up till now, only knew space of explicit formulas was contractible.

Class result for smooth CY:

Comm. analogue:

M supermanifold, $\gamma_1 \in T_{odd}(M)$, $[\gamma_1, \gamma_1] = 0$ homological v.f.

extend γ_1: weak CY • $(\gamma_2, \gamma_3, \ldots)$ polyvectorfields $\gamma_k \in \Gamma(\Lambda^k T(M))$

w/ $[\sum_{i\geq 1} \gamma_i, \sum \gamma_i] = 0$.

HC^-

Solve linear eq'n. • $(\omega_2, \omega_3, \ldots)$ $\omega_k \in \Gamma(\Lambda^k T^*)$. $(d + Lie_{\gamma_1})(\sum \omega) = 0$

~~3. condition γ_2 is non-deg. (up to homotopy)~~
~~ω_2 is non-deg.~~

Then, should give same stuff.
How to transfer between both?
Legendre transform identifies the M-C equations.

$$(\omega) \longleftrightarrow (\gamma)$$

$$\omega^2 \cdot \gamma^* = 1.$$

(quasi-iso. of A_∞ algebras w/ structures gives quasi-iso of weak CY algebras)

There should be a notion of nondegeneracy interpolating when smooth & proper

CC(A) non-unital.

~~TA⊗~~ $\left(TA^{cyc} \otimes TA[u] \right) [[u]]$

$$HH_* \xrightarrow{B} HC$$

~~HH_*(M)~~

$$HH_*(M) \otimes HH_*(F).$$

Y
↓
M

If algebra has involution $A \to A^{op}$, w/ M ..., $M^{op} = A$

$\mathcal{F}(M) \to \mathcal{F}(M^{op})$. $M^- \times M \to M \times M^-$
$\mathbb{Z}_2$.

$QH^*(\mathbb{P}^n) = \mathbb{C}[x,t] / x^{n+1} = t$ should split? (not as rings.)

guess? t odd ⊕ t even

$\mathrm{Ext}_{C^*(X)}(k, k) = C_* \Omega X$

~~Cobar ($C^*(X)$)~~

$C_*(X)^\vee = C^*(X)$

Cobar ($C_* X$)

$\simeq C_* \Omega_p X$.

$C_* X$ ok

$C_* X \to C_*(X) \otimes C_*(X)$

$[s] \mapsto$ split.

$B \Omega X \simeq X$

claim $C_* X$

"$B \Omega$" $C_*(X)$

$B C_* \Omega X \simeq C_* X$.

but $C_* \Omega X$ vs. Cobar $C_* X$

~~problem~~