Siebert

I. Review

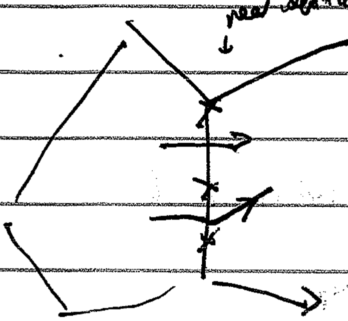
$B = \bigcup_{\sigma \in \mathcal{P}} \sigma$ "base of SYZ" $\quad \mathcal{X} \supset T^*B_0/\Lambda^*$

$\sigma \in \mathcal{P}$ ↑ $\mathbb{Z}$-polyhedra

$$\begin{array}{ccc} \mathcal{X} & \supset & T^*B_0/\Lambda^* \\ \downarrow & & \downarrow \\ B & \supset & B_0 = B \setminus \Delta \end{array} \qquad \operatorname{codim} \Delta = 2.$$

$\Delta \subseteq \mathcal{P}^{\text{codim } 2 \text{ bases}}$ of polyhedral str.

need identifications/gluings



$B = \bigcup_{\sigma \in \mathcal{P}} \sigma$

Aim: $\begin{array}{cc} X_s & \boxed{\mathfrak{X}} \\ \downarrow & \downarrow \\ s \in & \operatorname{Spec} A \end{array}$, degeneration. $\quad A = \mathbb{C}[[t]], \quad \mathbb{C}[t]/(t^{k+1})$

$\Lambda \subset T_{B_0}$ $\mathbb{Z}$-vectors

(Modern pov):

Construct $\mathfrak{X}^o = \mathfrak{X} \setminus \text{codim} 2$

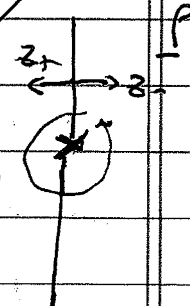
Local models for $\mathfrak{X}$: $\dim \sigma = n$ $\quad \mathcal{U}_\sigma = \operatorname{Spec} A[\Lambda_\sigma]$

~~$\dim \rho \geq ...$~~ $\qquad \cong \mathbb{G}_m^n / A$.

some primitive, unique gives z_-

$\dim \rho = \underline{n-1}$: $\quad \mathcal{U}_{\underline{\rho}} = \operatorname{Spec} A[\Lambda_\rho][z_+, z_-] / (z_+ z_- - f_\rho t^e)$

$\underline{\rho} \subset \rho \setminus \Delta$ connected component



$f_\rho \in A[\Lambda_\rho]$ ind. of z_+, z_-,

$t^e \in A$.

Just Glue, & have to have consistency conditions:

$$\mathfrak{X}^{o\prime} = \varinjlim_{\underline{\rho}, \sigma} (\mathcal{U}_\sigma \to \mathcal{U}_{\underline{\rho}}) \to \operatorname{Spec} A.$$

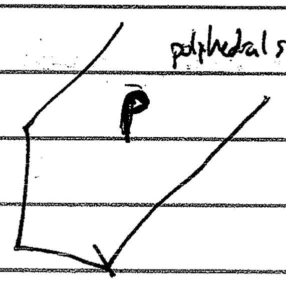
$B = \bigcup_{\sigma \in P} \sigma$, $\Delta \subset B$, f_ρ, $t^e \in A$

$U_\sigma = \mathrm{Spec}\, A[\Lambda_\sigma]$, $U_\rho = \mathrm{Spec}\, A[\Lambda_\rho][Z_+, Z_-]/(Z_+ Z_- - f_\rho t^e)$

Problem: (each piece may not have enough fcns.)

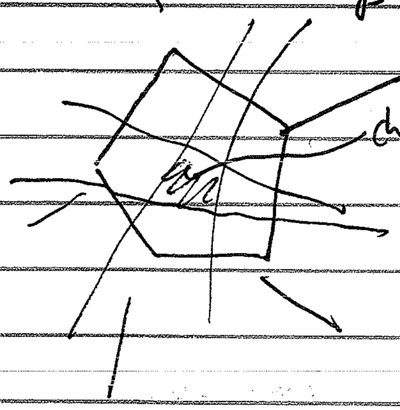
Need corrections to obtain local functions. $\rightsquigarrow$ wall structure.

Wall:

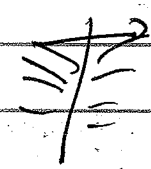


polyhedral subset $\mathfrak{p} \subset \sigma$, $\cap \operatorname{Int} \sigma \neq \emptyset$.

$g_\mathfrak{p}$



chambers. u: get $U_u = U_\sigma$ - when,



cross chamber

UCF: $z^m \longmapsto g_\mathfrak{p}^{\langle n_\mathfrak{p}, m\rangle} z^m$

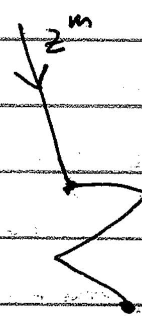
$\Lambda_\mathfrak{p}^\perp = \mathbb{Z} n_\mathfrak{p}$.

Now, $\mathring{\mathfrak{X}} := \varinjlim_{u,\rho} (U_u \to U_\rho)$.

Have $\{\mathfrak{p}\}$ (wall str.) $\rightsquigarrow \mathring{\mathfrak{X}} := \lim_{u,\rho} (U_u \to U_\rho)$.

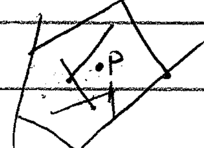
Canonical functions on $\mathring{\mathfrak{X}}$: $\vartheta_m(p) = \sum_{\text{broken lines } \beta} a_\beta z^{m_\beta}$

consistency $\rightsquigarrow$ well-defined



z^m

m: asymptotic monomial $=$ v.f. on B.

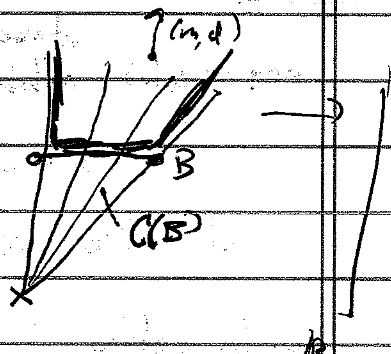


p

Rem: $\mathring{\mathfrak{X}}$ has ample line bundle

$$R = \bigoplus_{\substack{m \text{ asymp.} \\ \text{monomial}}} A \cdot \vartheta_m = \Gamma(\mathring{\mathfrak{X}}, \mathcal{O}_{\mathring{\mathfrak{X}}})$$

To get $\mathfrak{X} = \mathrm{Proj}(S)$: $\mathrm{Tot}(\mathring{\mathcal{L}}^{-1})$ $C(B)$,



$$S := \Gamma(\mathrm{Tot}(\mathring{\mathcal{L}}^{-1}), \mathcal{O}) = \bigoplus_{d \geq 0} \underbrace{\bigoplus_{\substack{m \text{ of} \\ \deg. d, \text{ i.e.} \\ m \in B(\frac{1}{d})}} A \cdot \vartheta_{(dm, d)}}_{= S_d}$$

II Asymptotic monomials / LG-models

$$S_0 = \bigoplus A \cdot \vartheta_m = R$$

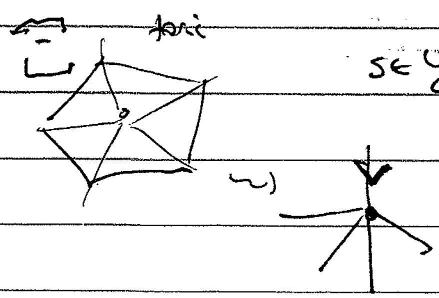
$d=0$ case

(If non-compact, paint left & right above)

$$\mathfrak{X} = \mathrm{Proj}\, S$$
$$\downarrow \text{proper}$$
$$\mathcal{Y} = \mathrm{Spec}\, S_0$$

LG: 1) Hori-Vafa asymptotically toric

tori

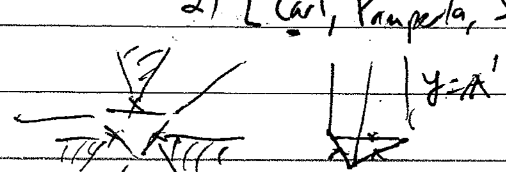


$\mathfrak{X} \equiv \mathcal{Y}$. $\mathcal{Y} \xrightarrow{W} \mathbb{A}^1$, and can vary coeffs,

$s \in \mathcal{Y}$ varies coeffs. of LG-potential)

$$W = \sum a_m z^m$$

2) [Carl, Pumperla, S.]: B asymptotically parallel.



$\mathcal{Y} = \mathbb{A}^1$ then $\mathcal{Y} = \mathbb{A}^1$, and $X \xrightarrow{W} \mathcal{Y} = \mathbb{A}^1$ LG potential.

III) Smoothing of conifolds (w/ H. Ruddat)

(Matessi, Castaño-B.)

Conifold: $(xy = zw) \subset \mathbb{C}^4$.

f_ρ [$\longleftrightarrow$ log structure.] simultaneously smooth conifold points

$\downarrow$ (resolution & smoothing is mirror)

$f_\rho(s)$, s parameter.

Existence of $\mathfrak{X}$ w/ conifold points:

partly no problem, partly ok w/ some work.

Claim: $\exists\ f_\rho(s)$ ~~$\leftrightarrow H^1(B\setminus\{p_i\}$~~

$\Updownarrow$

(*) $H^1(B\setminus\{p_i\}, i_*\Lambda^*) \longrightarrow H^2_{\{p_i\}}(B, i_*\Lambda^*) = \bigoplus_{\{p_i\}} \mathbb{C}^{(1/2)}$ has good image.

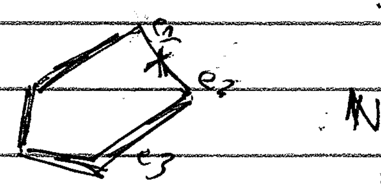
Friedman/Tian: $\exists$ simultaneous smooth $\Longleftrightarrow$ exc. curves of small resolution fulfill good relation.

In progress: believe (*) can be related to

(try to run resolution process here too ...)

(also prove Morrison: smoothing & resolution mirror)

(IV) Variation of Kähler



$(e_\rho) \longleftrightarrow$ target of MPL function.

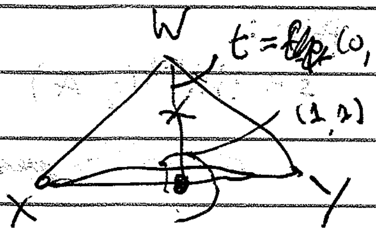
replace N by monoid Q; now $e_\rho \in Q$

$q: Q \to A$ monoid homomorphism (log structure)

makes ~~sense~~ the express t^{e_ρ} makes sense. (Q could map to some invertible elts)

Example $Q = \mathbb{N}^2$, $A = \mathcal{O}[s, s^{-1}, t]$

$(a,b) \longmapsto s^a t^b$



$f_p \rightsquigarrow (1 + s\frac{z}{w})$

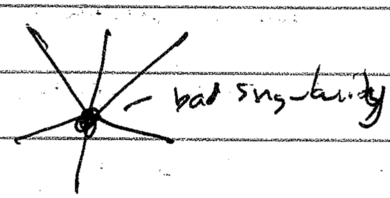
$f_p \rightsquigarrow (1 + s^{-1}\frac{w}{z})$

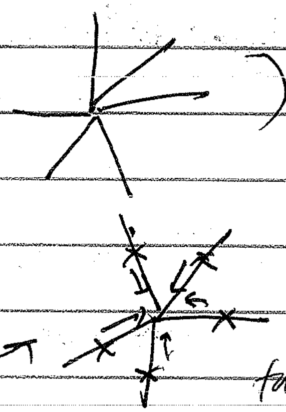
Exercise: $\leadsto \boxed{xy = (sz + w)t}$

interpolates between toric ~~...~~ and another.

Ex 2: surface singularities

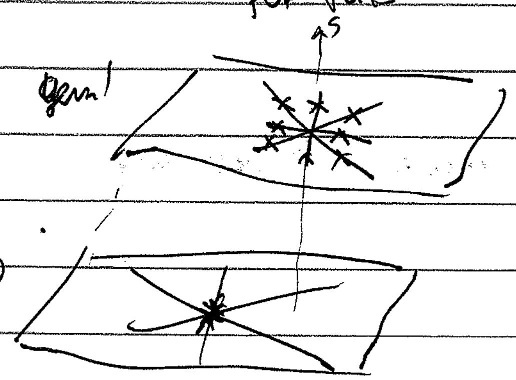
($\mathfrak{X}$ affine $\iff$ B conical)





family of simple singularities

Put these in a family:



$\tilde{B}$ family

$\wr$

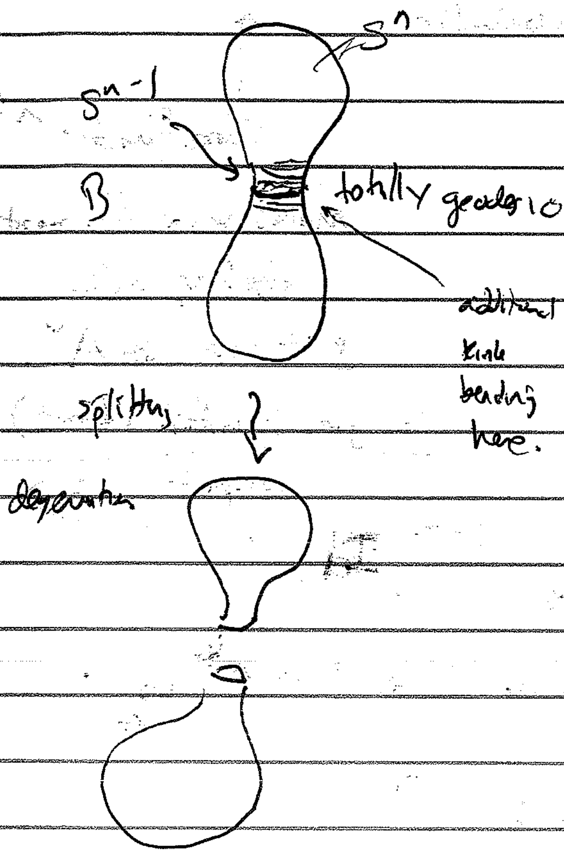
$\tilde{X} \circlearrowleft \mathbb{G}_m$

$\cup$

$(s=0) = X$ GHK singularity

Now, singularities have nice codim.

Type II - degeneration



$\mathcal{X} \longrightarrow \mathrm{Spec}\,(\mathbb{C}[s][[t]]) \longrightarrow \mathbb{A}^1_s$

$\mathcal{X}_1 \cup_{\mathcal{Y}} \mathcal{X}_2 \quad s=0$, $s=1$ generic (get Fano stuff & boundary divisor.)

mirror dual

Stretching the neck!