Abouzaid, On Family Floer Cohomology

building on 10-year old ideas of • Kontsevich-Soibelman & • Fukaya

A $\quad$ X $\quad$ Lagn torus fibration $\quad$ | $\quad$ B

$\quad\quad\downarrow$ $\quad\quad$ ⇝ SYZ $\quad$ Build a mirror space Y (rigid analytic), & prove HMS

$\quad\quad Q$ $\quad\quad\quad$ |

Family Floer homology should define the mirror functor.

Progress:

- **Fukaya:** Constructed the local patches of Y (in the presence of hol. discs)
- **Tu:** Fukaya's construction patches can be glued to a space $Y_{\emptyset}$ parametrizing the smooth fibers.
- **Fukaya:** (slides on webpage) If X K3 surface (usual Lagn torus fibration), can construct the entire mirror in this way.

Today: Toy case when $\begin{array}{c} X \\ \downarrow \pi \\ Q \end{array}$ has no singularities

(should be a general theory)

Q: what to do on the right hand side??

have $\mathcal{F}_\beta(X)$ $\forall$ $\beta \in H^2(X;\mathbb{Z}_2)$

Assumptions: ① Q is orientable

(If Q is not spin, the functor starts in $\mathcal{F}_{\pi^* w_2(Q)}(X)$ ← stiefel-whitney class)

On a sympl. manifold

② $\pi_2(Q) = 0$ ⇒ no instanton corrections on closed fibers (no hol. disc w/ bd. in π).

Q is an integral affine manifold; i.e. the $P \subset \mathbb{R}^n$ ← integral affine polyhedron ③ Q closed.

glued by transformations in $SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$.

$$\begin{tikzcd} Y_P \arrow[d] \arrow[r, phantom, "\subset"] & (\Lambda^*)^n \arrow[d, "\mathrm{val}"] \\ P \arrow[r, phantom, "\subseteq"] & \mathbb{R}^n \end{tikzcd}$$

$$\Lambda = \text{Novikov field}$$
$$= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; \begin{array}{l} \lambda_i \in \mathbb{R} \\ a_i \in \mathbb{k} \end{array} \right\}$$
$\lim_{i \to \infty} \lambda_i = +\infty$

$$\Lambda^* := \Lambda \setminus \{0\}$$

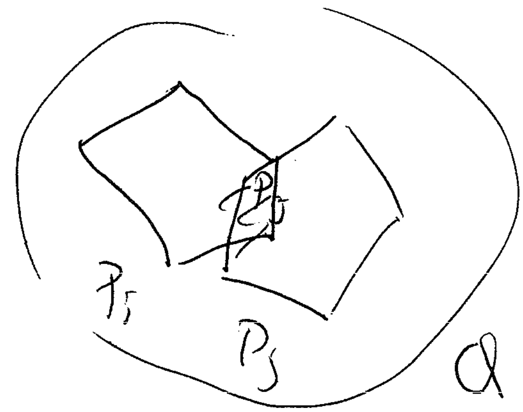
$$\mathrm{val}\left(\sum a_i T^{\lambda_i}\right) = \lambda_0 \text{ if } \lambda_0 < \lambda_1 \cdots$$

Y_P is an example of an "affinoid domain" which is studied by Tate.

Given a description of Q as

$$\bigcup P_{ij} \to P_i \sim Q,$$

obtain Y by gluing Y_{P_i}'s.



Define $$Y = \left(\bigcup_{P_{ij}} Y_{P_{ij}} \longrightarrow \bigcup_{P_i} Y_{P_i} \right)$$

<u>Claim:</u> Y parametrized pairs

$$\begin{tikzcd} \pi : X \arrow[d] \\ Q \end{tikzcd}$$

$(F, b) \mapsto$ class in $H^1(F, U_\Lambda)$

fiber of π

units of Λ

$$U_\Lambda \subset \Lambda^* := \mathrm{val}^{-1}(0)$$
$$= \{ a_0 + \sum a_i T^{\lambda_i},\ \lambda_i > 0 \}$$
$a_0 \neq 0$

this is like a map $\pi_1(F) \to U_\Lambda$

Λ local system on F w/ monodromy in U_Λ.

(a_0 is a $\mathbb{C}$-local sys., & rest is a bounding co-chain).

(Rmk: Y depends only on Q itself!, but Q, w/ int. affine str knows a lot about X).

We can define $H^*(L, (F, b))$ whenever $L \subset X$ is Lagr. in X

Fix $p \in Q$.

Arnold–Liouville: There is a canonical identification

$$\begin{array}{c} T_p Q \cong H^1(F_p, \mathbb{R}) \\ \cup\!\! \mid \\ P_{\text{polygon}} \end{array}$$

$$\begin{array}{ccccc} H^1(F_p,\Lambda)\supset & Y_P & \subset & H^1(F_p,\Lambda^*) \\ & \downarrow & & \downarrow \\ p \in & P & \subset & H^1(F_p,\mathbb{R}) \end{array}$$

If p' is near p, we can identify it with a vector in T_pQ. If we trivialize $H^1(F_p,\mathbb{R})$, $\rightsquigarrow (v_1, \dots, v_n)$.

~~Identify~~ Embed

$$H^1(F_{p'}, U_\Lambda) \hookrightarrow H^1(F_p, \Lambda^*)$$

image set of elts. whose valuation is $(v_1, \dots, v_n)$.

Naive idea: $HF^k(L,(F,b))$ is the fibre of a coherent sheaf $H^k\mathcal{L}$ defined on Y.

The data of a coherent sheaf is equivalent to sheaves on Y_i together with gluing data.

($Y_i \rightsquigarrow$ Affine ..., a module over $\mathcal{O}_{Y_i}$,

Tate's affinoid ring: Formal Laurent series in $H_1(F,\mathbb{Z})$ which T-adically converge on Y_i).

$$\sum_{\vec{\alpha} \in \mathbb{Z}^n} c_{\vec{\alpha}} z^{\vec{\alpha}}, \quad c_\alpha \in \Lambda \quad z = (z_1, \dots, z_n)$$

converges for elts. of $Y_i \iff$ some decay/growth condition on c_α.

Near a point:

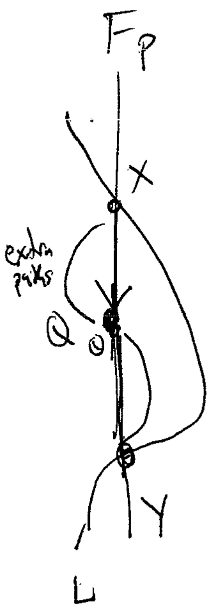
Fix $p \in Q$, a Lagrangian $L \subseteq X$. (a priori don't know L transverse to X_p)

Pick a Hamiltonian H w/ time 1-flow φ, s.t. $F_p \pitchfork \varphi(L)$

~~Given $p \in P \subset Q$~~ Usually, take $\bigoplus_{x \in F_p \cap \varphi(L)} \mathbb{K}\langle x \rangle$

Given $p \in P \subset Q$, instead consider $\bigoplus_{x \in F_p \cap \varphi(L)} \mathcal{O}_{Y_P}\langle x \rangle =: \mathcal{L}(p)$ free $\mathcal{O}_{Y_P}$ module.

Fix an origin for F_p, ~~Assign~~ and a path from origin to $x \in \varphi(L) \cap F_p$



$\rightsquigarrow$ Assign a class $[\partial u] \in H_1(F_p, \mathbb{Z})$

for every strip with boundary on L & F_p. $\}$ $\in \mathcal{M}(y,x)$

Define $$dx = \sum_y \Big(\underbrace{\sum_{u \in \mathcal{M}(y,x)} \pm T^{\int u^*\omega} z^{[\partial u]}}_{} \Big) \cdot y$$

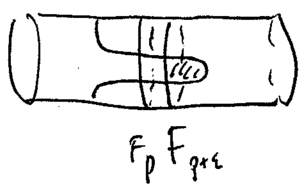
(Recall that $\mathcal{O}_P$ is a completion of Laurent polynomials on $H_1(F_p, \mathbb{Z})$)

Q: Is this an element of $\mathcal{O}_{Y_P}$? (~~equivalent~~ assuming plug in $z = 1$, this is usual convergence)

Usual story $\Rightarrow$ convergence for $z \in H^1(F_p, U_\Lambda)$.

Lemma: (Fukaya): If P is sufficiently small, then d converges. (trick i need to know you can do floer theory w/ tame almost cplx. structures.)

$\Rightarrow \mathcal{L}(P)$ is a complex.

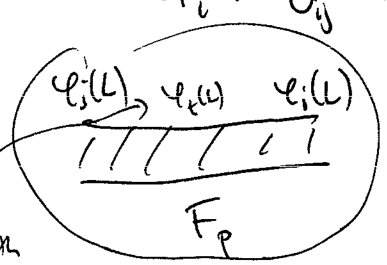


Fix a cover $P_i \ni p_i$ and $\varphi_i(L) \cap F_{p_i}$.

Goal: Construct a gluing map

$$\mathcal{L}(P_i) \otimes_{O_{ij}} O_{ij} \xrightarrow[\text{continuation}]{K} \mathcal{L}(P_j) \otimes_{O_j} O_{ij}.$$

$O_I := O_{Y_{P_I}}$



(have: $CF^*(\phi_i(L), F) \overset{q.i.}{\simeq} CF^*(\phi_j(L), F)$).

choose a path
continuation

Problem: Why does K converge?

Ans: it may not, but:

Lemma: (A.) If we refine the cover sufficiently, then K converges on P_{ij}.

Cocycle condition → Ensure convergence by refining further

→ Choices of basepoints.

<u>Thm</u>: (Gross, Duistermaat):

X is uniquely determined up to fibrewise symplectomorphism

by $\left(Q, \text{Integral affine structure}, [\alpha_X] \in H^2(Q, \mathrm{Aff})\right)$

Čech

$\exists \exp : H^2(Q, \mathrm{Aff}) \longrightarrow \check{H}^2(Y, \mathcal{O}^*)$

$\alpha_X \in \check{H}^2(Y, \mathcal{O}^*)$ failure of co-cycle condition

$\Rightarrow H \cdot \mathcal{L}$ is an α_X-twisted coherent sheaf.

$(A, H\!A.(A, A))$