stability condition ⇝ spectral curve of 3-D Hitchin system

-- spectral curves of Gaiotto - Neitzke.

Mean curvature flow --

(If you use solutions of M-C by mean curvature, get negative powers of variables, use Bridgeland asymptotics to decompose. ~).

($L_1 \subset L_2$, get embedding of objects).

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Auroux, HMS for hypersurfaces in $(\mathbb{C}^*)^n$ (w/ Abouzaid) in progress

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1205.0053

Geometric setup: (Abouzaid - A - Katzarkov)

(see also Clarke, G-K-R, ~)

$$H = \{ f_t = \sum_{\alpha \in A} c_\alpha t^{\rho(\alpha)} x^\alpha = 0 \} \subset (\mathbb{C}^*)^n$$

$A \subseteq \mathbb{Z}^n$ $\quad x_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ (family) of hypersurfaces.

$\rho : A \to \mathbb{R}$, $\quad t$ formal $|t| \ll 1$.

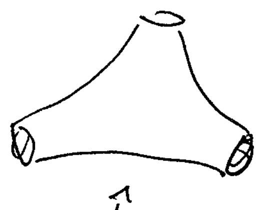
⇝ Construct s/Z mirror to $H$, $(Y, W)$ $\forall$ toric $C$-$Y$ $(n+1)$-folds, moment polytope $\Delta_Y = \{ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R} \mid \eta \geq \varphi(\xi_1, \dots, \xi_n) \}$.

$\varphi = \mathrm{Trop}\, F = \max_{\alpha \in A} ( \langle \alpha, \xi \rangle - \rho(\alpha) )$.

and $W = -z^{(0,\dots,0,1)}$

toric monomial.

Key example: $(n-1)$-dimensional pair of pants $H = \{x_1 + \dots + x_n + 1 = 0\} \subseteq (\mathbb{C}^*)^n$.



$\updownarrow$

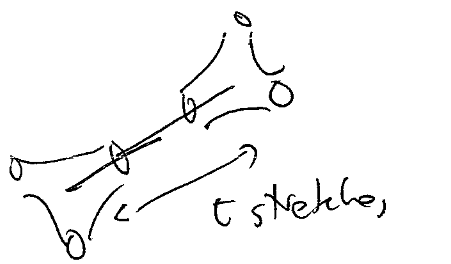
$\mathbb{C}^3, -z_0 z_1 z_2$

$\updownarrow$

$(Y = \mathbb{C}^{n+1}, W = -z_0 \cdots z_n)$

(all powers of $t$ are zero; don't need to put any power of $t$ to tropicalize).

(no $t$ b/c no moduli: e.g. 4-punctured $\rightsquigarrow$




)

$H$ doesn't seem to carry a suitable Lagr. $T^{n-1}$ fibrations.

but:

1) $\mathcal{F}(H) \underset{(\simeq ?)}{\hookrightarrow} \mathcal{F}(X, W_X; s)$

$X$ = blow-up of $(\mathbb{C}^*)^n \times \mathbb{C}$ at $H \times 0$ (twisting cocycle.)

$W_X$ = lift of coordinate on $\mathbb{C}$, $s \in H^2(X, \mathbb{Z}/2\mathbb{Z})$ changes signs of discs

$W_X$ has Morse-Bott singularities w/ critical locus $\mathrm{crit}(W_X) \simeq H$.

$s$ accounts for twisting of normal bdles along critical locus.

2) $(X, W_X, s) \xleftrightarrow{SYZ} (Y, W)$ (see also Chan-Leung-Lau) for other direction from $(Y,W)$

i.e. $\exists\ \pi : X^\circ \longrightarrow B$ Lagr. $T^{n+1}$-fibration, s.t. $Y$ = completed moduli space of ~~these objects~~ $T^{n+1}$-objects of $\mathcal{F}(X^\circ)$ supports on fibres of $\pi$

& in $\mathcal{F}(X, W_X, s)$, these objects become weakly unobstructed in the sense of FOOO w/ $m_0 = W$.

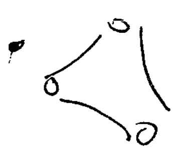
(in this case $X^\circ = X \setminus D$, $D$ = proper transform of $(\mathbb{C}^*)^n \times 0$)

Two directions of HMS

1) wrapped Fukaya category ~~$\mathcal{F}$~~

$$\mathcal{W}(H) \simeq D^b_{sing}(Y, W)$$

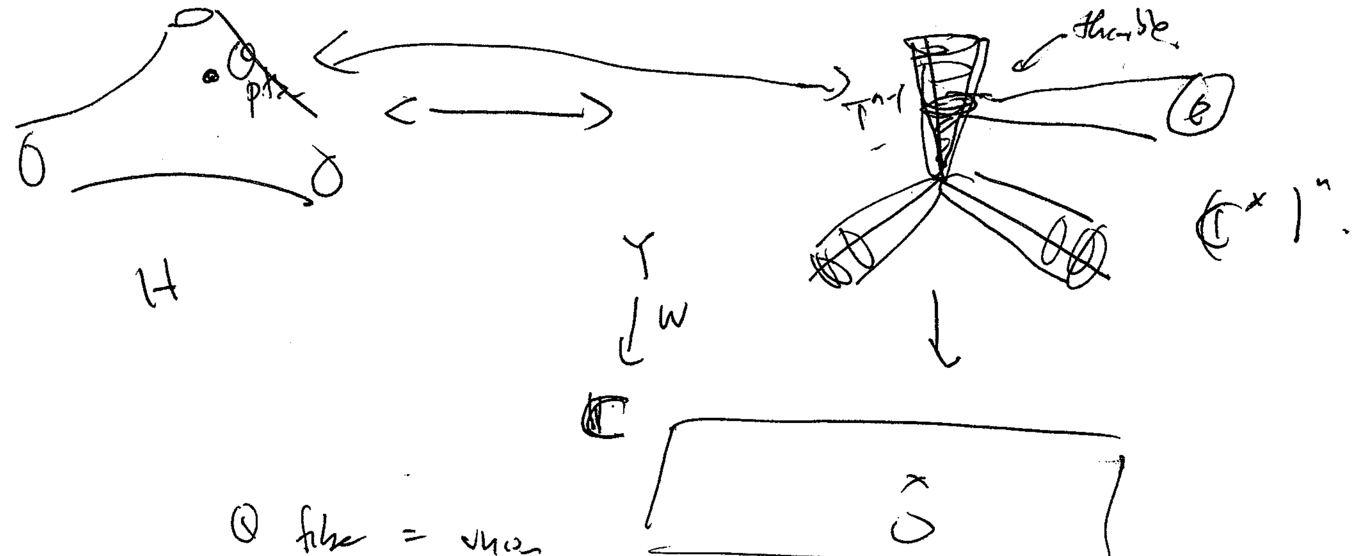
done in several cases:

-  & ~~AA~~ AAEKO 2011
- Riem. surfaces $\subseteq (\mathbb{C}^*)^2$ Bocklandt
  H. Lee in progress
- higher dimension pairs of pants, for compact Fukaya category, Sheridan.

Here: $D^b Coh(H) \underset{?}{\simeq} \mathcal{W}(Y, W)$.

fiberwise wrapped

e.g.




0 fiber = union
of toric strata,

generic fibers are $(\mathbb{C}^*)^n$'s.

Get 3 cylinders worth of objects, that
somehow glue together to get a pair of pants
worth of objects.

(tracking is in bounding co-chains).

Want more than pants worth

$\mathcal{O}_H \rightsquigarrow$ it generates category, & $\mathrm{End}(\mathcal{O}_H)$ are just
fns on $H$.

$L_0$ ??

idea that doesn't work: take real positive skeleton
can build singular Lag'ns, but no idea how to do
Floer theory??

~~Main results~~

Main results (Abouzaid - A)

1) Can define in this setting a fibrewise wrapped Fukaya category $\mathcal{W}(Y, W)$

[ad hoc, using restrictive assumptions on what objects we allow
... may not be general enough to get objects in non-toric case]

Z - Sylvan in progress

2) Using this def'n, can construct an object

$L_0 \in \mathcal{W}(Y, W)$ (conj. it generates).

↑ but not much pt. trying to prove it unless $\mathcal{W}(Y,W)$ is enlarged...)

3) Calculate $\mathrm{End}(L_0) \underset{A_\infty\text{-alg}}{\simeq} \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \Big/ \left(\sum_\alpha c_\alpha t^{P(\alpha)} x^\alpha = 0\right)$

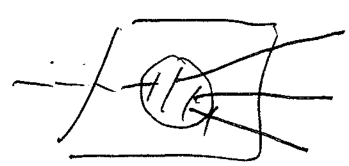
(so, formal).

$\simeq \mathrm{End}(\mathcal{O}_H)$

[hence, ~~$D^b Coh(\check{Y}, W)$~~ $\subset$
$D^b Coh(H) \hookrightarrow \mathcal{W}(Y, W)$]

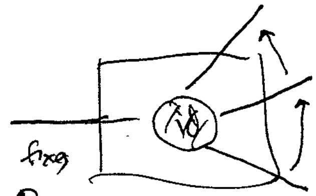
1) $\mathcal{W}(Y, W)$? Also need $\omega$ of course.

Objects: properly embedded Lagr. submanifolds $L \subset Y$
(+ spin str. ...)
s.t. $W(L) \subset \mathbb{C}$



finite union of rays $\neq \mathbb{R}_{<0}$

• $\rho_t$ flow on $\mathbb{C}$



fixing $\mathbb{R}_{<\infty}$.

lift $\rho^t(L)$ using parallel transport.

by parallel transport, Lag'n.

- $\phi^t$ = flow of a Hamiltonian $H : Y \longrightarrow \mathbb{R}$

"linear wrapping Hamiltonian" (toric case: fn. of moment map coords.)

invt. under parallel transport, and fibrewise proper (so actually ~~commutes~~ w/ $\rho_t$).

$$L^t \underset{\text{def}}{=} \phi^t \rho^t L.$$

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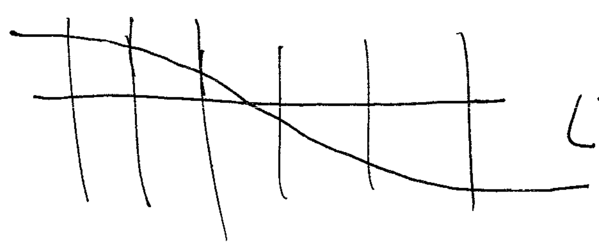
Technical conditions: • $\forall L, L'$, for an open dense set of $t$'s,

outside a cpct. subset, $d(L^t, L') > c > 0$ (not uniform in $t$).

($\longrightarrow$ ensures compactness for J-hol. curves).

- For $0 < t' - t \leq \varepsilon$, $L^{t'}$ vs. $L^t$




that goes to $\infty$ for different gradings

(makes things look linear wrapping)

- $L$ doesn't bound holom. discs.

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directed cat: objects $(L, k)$, $k \in \mathbb{Z}$

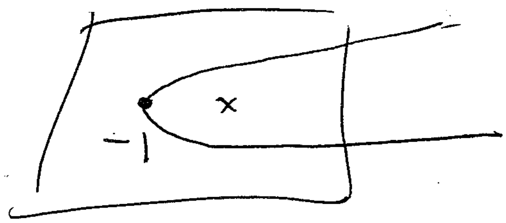
$$\hom((L,k),(L',k')) = \begin{cases} CF(L^{k\varepsilon}, L'^{k'\varepsilon}) & \text{if } k > k' \\ \mathbb{K}\cdot\mathrm{id} & k = k' \\ 0 & \text{otherwise.} \end{cases}$$

Localize w.r.t. "Id": $(L, k+1) \to (L, k)$.

2) $L_0$.

Note:

$Y_{\mathbb{R}>0} \xrightarrow{W} \mathbb{R}_{<0}$. e.g. $\mathbb{C}^3 \to \mathbb{C}$, $-z_0 z_1 z_2 = W$.




so take $(\mathbb{R}^+)^n \subset W^{-1}(-1)$,

& parallel transport to something going around singular fibre, to get $L_0$.

Facts: a well-chosen fibrewise wrapping $H$ keeps $L_0^t$ away from $L_0$ $\forall\, t \notin 2\pi\mathbb{Z}$. (using parallel transport in toric case)

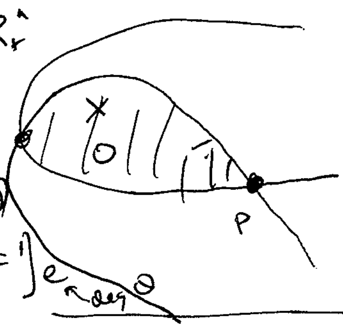
3) Calculate $\mathrm{End}(L_0)$ i.e. $CF(L_0^t, L_0)$ for $t \gg 0$.

$\mathbb{R}^n_+ \cap \phi_t \mathbb{R}^n_+$

$\cap\!|$

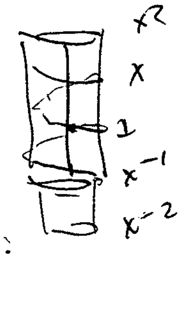
$(\mathbb{C}^*)^n$

gives us (truncated) $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] e_{n\deg 0}$




limit as $t$ goes <u>deformed</u>:

e.g. $W^{-1}(-1)$ when $n = 1$.



Over p, get also

(truncated) $\mathbb{K}[x_1^{\pm} \cdots x_n^{\pm}]h$

$\uparrow$ deg $-1$.

Differential is 0 on each block, but cross-term counts sections.

Need to analyze # of sections:

Calculation uses Seidel TQFT

+ trick in J. Pascaleff's thesis.

Deconstruct to study product structures

Understand what happens when you move singular fibre around.

pair of pants $(\mathbb{C}^{n+1}, -z_0 \cdots z_n)$

- ~~$\mu$~~ $\mu^1(h) = \left[\sum_{\alpha} 1 + x_1 + \cdots + x_n\right] e$

compute! $\to$

exactly defining equation.

linear over $\mathbb{K}[x_1^{\pm 1}]$

product $\mu^2(e,e) = e$.

$e \cdot h = h$

$h \cdot e = h$

$h \cdot h = 0$ for degree reasons.

$\mu^{\geq 3} = 0$ for basic geometric reasons

so, $HF^* = \mathbb{K}[x_1^{\pm 1}, \ldots] e / (1 + x_1 + \cdots + x_n)$.

In general case, GW theory of toric C-Y coming as corrections

$$u^2(h) = \sum_{\alpha} (1 + \cdots) t^{P(\alpha)} x^{\alpha}.$$

One family of hol. sections for each toric divisor in Y.

But there can be corrections coming from

can find formulas for this,

is Chan-Leung ---- .