Efimov, ~~N~~Noncommutative Hitchin System

$G = GL_n(\mathbb{C})$

X sm. proj. curve

$\mathcal{M}^s_{r,d}(X)$ $\qquad T^*\mathcal{M}^s_{r,d}$

↑ deg, rank stable bdles

$$\cap$$

$$(E, \varphi \in H^0(End(E) \otimes \omega_X))$$

$$\xi \in H^0(X, \omega_X^{\otimes n})^*$$

$$\xi(E, \varphi) = \xi(Tr(\varphi^n)).$$

$$Sym\left(\bigoplus_{i=1}^{r} H^0(X, \omega_X^{\otimes n})^*\right) \hookrightarrow Fun(T^*\mathcal{M}^s_{r,d})$$

The image is Poisson commutative.

$$\dim \bigoplus_{i=1}^{n} H^0(X, \omega_X^{\otimes i})^* = r^2(g-1)+1 = \dim \mathcal{M}^s_{r,d}.$$

Claim: Hitchin system for GL_r has NC categorical analog –

$$\mathcal{J}_\mathcal{E} := D^b(X) / \langle \mathcal{E} \rangle$$

quotient by sub-cat. gen. by simple vec. bdle

Assume $End_X(\mathcal{E}) = \mathbb{C}$

$\ne 0$.

Easy to understand this quotient category: fix $p \in X$.

(1) Then, image of $\mathcal{O}_p$ is a split-generator of $\mathcal{J}_\mathcal{E}$

(2) $REnd_{\mathcal{J}_\mathcal{E}}(\mathcal{O}_p)$ is just an associative algebra, (only 0 cohomology). call it B_p.

(e.g. use Drinfeld quotient to see

$$\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{E}) \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{O}_p) \to \mathrm{Ext}^2(\mathcal{O}_p, \mathcal{O}_p)$$

mult. kills Ext^1)).

Properties:

I) Each $\mathcal{B}_{\mathcal{E}}$ is homotopically finitely presented (essentially by Töen)

(e.g., finitely <u>presented</u> and homologically smooth).

(not evident from definition).

II) $\mathcal{B}_{\mathcal{E}}$ is actually <u>Quiller smooth</u> (not just hfp),

(<u>Def</u>: A assoc. algebra, A <u>Quiller smooth</u> if

Ω_A - projective over $A \otimes A^{op}$.

$\uparrow$ bimodule of differentials

(recall: $0 \to \Omega_A \to A \otimes A \to A \to 0$)

Ω_A proj $\Rightarrow$ proj. dimension of A at most 1.

$\Updownarrow$

lifting property for sq. 0 extensions:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed, "\exists"'] & B \\ & \tilde{B} \arrow[u] \end{tikzcd}$$

sq. 0 extension)

<u>Pf</u>: write: $D^b(X) = \mathrm{Perf}(\mathcal{A}_X)$

$\mathcal{A}_X = R\mathrm{End}(\mathcal{E} \oplus \mathcal{O}_p)$.

Then have morphism $\mathcal{A}_X \to \mathcal{B}_{\mathcal{E}}$.

To show Quiller smooth: need $\mathrm{Ext}^2_{\mathcal{B}_{\mathcal{E}} \otimes \mathcal{B}_{\mathcal{E}}^{op}}(\mathcal{B}_{\mathcal{E}}, \mathcal{B}_{\mathcal{E}} \otimes \mathcal{B}_{\mathcal{E}}^{op}) = 0$ each sheaf on $X \times X$ shifted by 1 ...

$\Delta_* \mathcal{O}_X$.

$$\parallel$$

$\mathrm{Ext}^2_{\mathcal{A}_X \otimes \mathcal{A}_X^{op}}(\mathcal{A}_X, \mathcal{B}_{\mathcal{E}} \otimes \mathcal{B}_{\mathcal{E}}^{op})$

dim. 2

which is 0 by dimension reasons on $X \times X$, 2 shift by 1 = Ext^3.

~~Azqui~~

General story:

A Quillen-smooth $\rightsquigarrow$ $\mathrm{Rep}_V(A)$ are smooth affine schemes.

V f.d. vector space.

$\mathrm{Rep}_V(A)$ - affine scheme w/

$$\mathrm{Rep}_V(A)(R) = \mathrm{Hom}_{alg}(A, R \otimes \mathrm{End}(V))$$

Get a natural $GL(V)$ action on $\mathrm{Rep}_V(A)$.

Interested in finite dimensional modules over $B_{\mathcal{E}}$.. $\mathrm{mod}_{f.d.}\text{-}B_{\mathcal{E}}$

Prop: If $\mathcal{E}$ is unstable, then $\Rightarrow$ $\mathrm{mod}_{f.d.\text{-}B_{\mathcal{E}}} = 0$.

why? ps-perfect modules $(B_{\mathcal{E}}) = \mathcal{E}^{\perp} \subset D^b(X)$
↑ pseudo perfect

If $\mathcal{E}$ unstable, then $\Rightarrow$ $\mathcal{E}^{\perp} = 0$

Assume $\mathcal{F} \in \mathcal{E}^{\perp}$ v.b.

(Pf: If $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ $\Rightarrow$ $\mathrm{slope}\, \mathcal{F} = \mathrm{slope}(\mathcal{E}) + g - 1$

Riemann-Roch.

But if unstable, have map ~~$\mathcal{E} \to \mathcal{E}'$~~
~~slope $\mathcal{E}'$~~ $\mathcal{E} \twoheadrightarrow \mathcal{E}'$, w/ $\mu(\mathcal{E}') < \mu(\mathcal{E})$

$\chi(\mathcal{E}', \mathcal{F}) > 0 \Rightarrow \mathrm{Hom}(\mathcal{E}', \mathcal{F}) \neq 0 \Rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0.$).

Have $SS(\mu) \subset \mathrm{Coh}(X)$
category of semi-stable vector bundles with slope μ.

$SS(\mu)$ - abelian of finite length, & simple objects correspond to stable bundles

It's clear that

(b) $\mathrm{mod}_{f.d.} B_{\varepsilon} \subset SS(\mathfrak{n}+\mathfrak{g} - \mathbb{A})$

$\uparrow$ Serre subcategory. (Q: how do you see functor takes pure objects to pure objects?

<u>Ans.</u> using perverse t-structures —).

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standard calculus for smooth algebras:

A — Quillen-smooth algebra.

$$\Omega^n_A := (\Omega_A)^{\otimes n}_A$$ nth tensor power over A. also a superalgebra

de Rham differential

$$d : \Omega^n_A \longrightarrow \Omega^{n+1}_A$$

first: $d : A \longrightarrow \Omega^1_A$.

$$d(a) = a \otimes 1 - 1 \otimes a.$$

Then note each elt. of Ω^n_A can be given as $a_0 da_1 \cdots da_n$.

Not interesting: one cohomology in deg 0. w/ $d(da_i) = 0$.

<u>Next</u>:

$$DR^\bullet(A) = \Omega^\bullet_A / [\Omega^\bullet_A, \Omega^\bullet_A]$$

Get $d : DR^n A \longrightarrow DR^{n+1} A$.

with representation spaces, get a map of complexes:

$$(DR^\bullet A, d) \longrightarrow (\Omega^\bullet \mathrm{Rep}_V(A), d).$$

Given the usual map

$$A \to \mathbb{C}[\mathrm{Rep}_V(A)] \otimes \mathrm{End}(V)$$, then take trace.

get $HH_0(A) \to \mathbb{C}[\mathrm{Rep}_V(A)]$

$$\begin{array}{c} HH_0(A) \\ \| \\ A/[A,A]. \end{array}$$

<u>Rmk:</u> $\mathrm{Der}(A,M) = \mathrm{Hom}_{A \otimes A^{op}}(\Omega_A, M)$

$\mathrm{Der}(A,A)$

standard calculus:

Given $\Theta \in \mathrm{Der}_{\mathbb{C}} A$, substitution gives

$$i_\Theta : DR^n A \to DR^{n-1} A,$$ & Lie der., std. relations.

A symplectic structure on A is a closed 2-form $\omega \in DR^2(A)$, $d\omega = 0$, giving an isomorphism

$$\mathrm{Der}_A \to DR^1 A.$$

If (A,ω) symplectic, then (A needs to be smooth commut.)

$(\mathrm{Rep}_V(A), \mathrm{Tr}\,\hat{\omega})$ — symplectic affine variety.

In this situation, $A/[A,A]$ is a Lie algebra.

$A/[A,A] \to \mathbb{C}[\mathrm{Rep}_V(A)]$ morphism of Lie algebras

$a \in A \rightsquigarrow da \in DR^1 A \rightsquigarrow \Theta_a \in \mathrm{Der}_A.$

$\{a,b\} = \Theta_a(b).$

$$\mathbb{D}er_A = Der(A, A\otimes A) = Hom_{A\otimes A^{op}}(\Omega_A, A\otimes A)$$

Der. bimodule

projective f.g. bimodule (as Ω_A is).

By definition,

$$T^*A := T_A(\mathbb{D}er_A)$$

tensor algebra over A of this bimodule.

From this: $0 \to T^*A \otimes_A \Omega_A \otimes T^*A \to \Omega_{T^*A} \to T^*A \otimes_A \mathbb{D}er_A \otimes_A T^*A \to 0$.

it implies that if one has:

$$\begin{array}{ccc} \mathbb{D}er_A \otimes_{A\otimes A^{op}} \Omega_A & \to & DR^1 T^*A \\ \| & & \cup \\ Hom_{A\otimes A^{op}}(\Omega_A, \Omega_A) \ni id & \mapsto & \lambda \end{array}$$

Liouville one form,

and $(T^*A, d\lambda)$ is symplectic noncommutative variety.

$f : A \to End(V)$

$T_f \, Rep_V(A) = Der(A, End(V))$

$= Hom_{A\otimes A^{op}}(\Omega A, End(V))$

$T_f^* Rep_V(A) = Hom_{A\otimes A^{op}}(\mathbb{D}er_A, End(V))$

$Rep_V(T^*A) = T^* Rep_V(A)$