Goncharov : Open-string Hodge theory

X ~~proj~~ var.

k ↙ ↘ $\mathbb{C}$

Étale

- $\mathrm{Gal}(\bar{k}/k)$ acts on
- $H^i_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)$
- étale site ⇒

étale sheaves --

Hodge

$H^*(X, \mathbb{C}) \xleftarrow{\text{Deligne}}$ MHS on. (using Hodge, hom. alg, & res. of singularities)

$G_{Hod,\mathbb{R}}$

? Hodge site

(really have $\mathcal{X} \downarrow B$, theory should explain why direct images have e.g. VHS).

Hodge: X smooth proj $\Rightarrow$ Hodge str. on $H^k(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$

"Brownian" path | Laplace

$\mathbb{R}$-MHS := Rep(pro-alg. group = Hodge group G_{Hod})

Hodge site [w/ Arbot '11.] := Holonomic D-modules on X (w/ irregular singularities), plus ~~an A_∞-autoequiv~~ homotopy-action of $G^{\mathbb{R}}_{Hod}$ (analogue of Galois gp.) by A_∞ autoequiv. of this A_∞ (really, dg) category.

More: open string structure on $\mathcal{D}_{hol}(X)$.

Today: $\mathcal{D}_{sm}(X)$ regular, and situation where $\mathcal{X} \downarrow B$ fibers cpct. & smooth, & real in B construction.

Output: genus 0 of OSHT $\Longleftrightarrow$ A_∞-action of G_{Hod}

$B = *$

genus > 0 "quantum version"

1) Preparation: "Surface BV algebra" of the category $CS^{\circ}(-"-)$.

$\mathrm{Har}_{\mathfrak{X}/\mathcal{B}}$ ← ~~Harmonic bundles~~"

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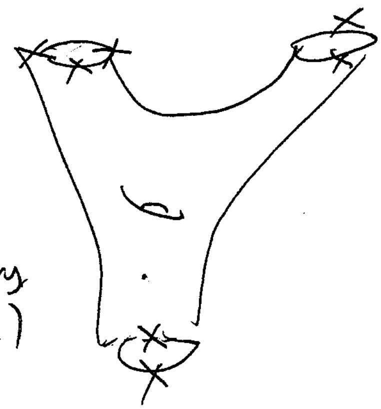
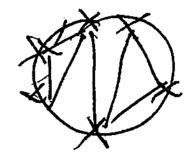
1) Decorated surface



S

- S: an oriented top surface w/ special pts. on the boundary (every bd. has special surface)

Simplest S - 

- Ideal triangulation T of S
  - a triangulation w/ vertices at special points.

$$\mathcal{M}_S := \mathcal{T}_S \big/ \Gamma_S$$

$\mathcal{T}_S$ ← Teichmüller sp.

Γ_S ← mapping class gp.

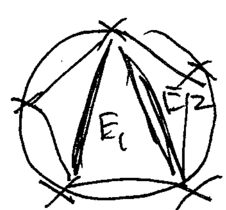
↑ moduli of hyperbolic str. on S.

Given a triangulation T, one gets ⟶

$$i_T : \mathcal{T}_S \longrightarrow \mathbb{R}_{>0}^{\{\text{internal edges of } T\}}$$

coordinates,

Γ_S - equivariant



e.g. $\mathcal{T}_{\bigcirc} \simeq \mathrm{conf}_{\cdot, m}(\mathbb{RP}^1)$

X_{E_1}, X_{E_2}. defined as some sort of cross ratio.

Most importantly,

$\exists\ \mathrm{vol}_S := d\log X_{E_1} \wedge \cdots \wedge d\log X_{E_n}$

(different triangulation: volume ~~form~~ same up to sign).

$\mathcal{A}^\circ$ ~~...~~ – a graded semisimple category, $2n$-CY category.

meaning: $\mathcal{A}^\circ(X,Y)\otimes\mathcal{A}^\circ(Y,X)\longrightarrow H$ ← line, deg $+\underline{2n}$.

non-deg. pairing.

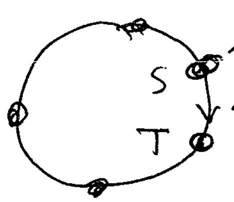
Har_X – category of harmonic bundles (Carlos Simpson)

<u>objects</u> are semi-simple local systems on X.

$$\mathrm{Hom}^\circ(\mathcal{L}_1,\mathcal{L}_2) := H^\circ(X, \mathcal{L}_1^*\otimes\mathcal{L}_2) \qquad H := H^{2n}(X)$$

(· useful / computable b/c of theorems of Simpson, Donaldson, etc ...)

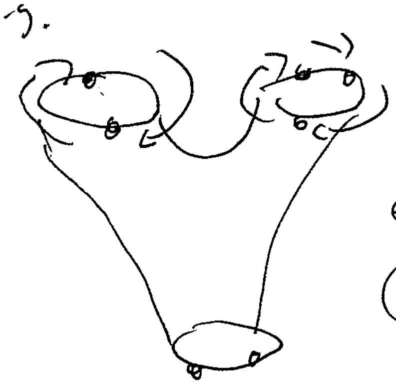
Given $\mathcal{S}$:



$\rightsquigarrow X_s \in \mathrm{ob}\,\mathcal{A}$

edge $\rightsquigarrow$ $\mathcal{A}_\circ(X_s, X_t) := \mathcal{A}^\circ(X_s,X_t)^\vee$

e.g.



e.g. on boundary component,

$$(\mathcal{A}_\circ(X_{s_1},X_{s_2})\otimes\mathcal{A}_\circ(X_{s_2}\otimes\cdots\otimes\mathcal{A}(X_{s_k},X_{s_1})))_{\mathcal{C}}$$ ← coinvariants w.r.t. the cyclic grp (which <u>is</u> the mapping class gp. in this case)

↑ think of this as an element of the <u>cyclic</u> tensor product

$$CT(--) = T(-)/[\,,\,]$$

For a surface $\mathcal{S}$, cat. $\mathcal{A}$, can associate

$$C^\bullet(\mathcal{S}, \mathcal{A}) := \Big(\bigotimes_{i=1}^{k} C^\bullet(c_i, \mathcal{A})\Big)_{S_n}.$$

all boundary components.

Euler char. of $\mathcal{S}$

Def: $$CS^\bullet(\mathcal{A}) := \bigoplus_{[\mathcal{S}]} C^\bullet(\mathcal{S}, \mathcal{A}) \otimes H^{\chi_{\mathcal{S}}} \otimes \mathrm{vol}_{\mathcal{S}}$$

can be disconnected.

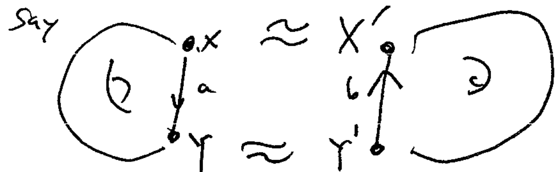
line in degree $\dim \mathcal{M}_{\mathcal{S}}$.

Clearly a commutative algebra

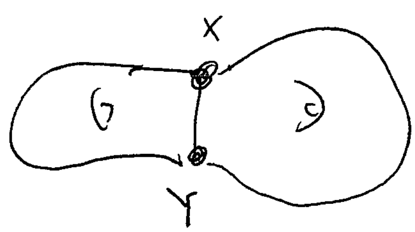
using $\mathcal{S}_1 \cup \mathcal{S}_2$.

$\exists$ BV Laplacian Δ (related to work of Costello/Sullivan/Zwiebach ...).

say



then can glue:



get some operation:

has degree 1.

$$\gamma_{a\sim b} : CS^\bullet(\mathcal{S}, \mathcal{A}) \to CS^\bullet(\mathcal{S}/_{a\sim b}, \mathcal{A})[1]$$

How to get it? Insert

$$\langle * \cap H \cap * \rangle\, d\log X_E$$

Insert intersection pairing. $\quad$ intersection pairing. $\quad$ deg. $+1$

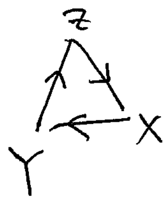
even line, can move it

$$\langle * \cap H \cap * \rangle : \mathcal{A}.(X,Y) \otimes \mathcal{A}.(Y,X) \otimes H \to \mathbb{C}$$

$$\Delta := \sum_{\text{all possible gluings}} \gamma_{a\sim b}. \quad (\Delta^2 = 0 \text{ bc of } d\log X_E \text{ is give a sign.})$$

$\mathcal{S} \longrightarrow \mathcal{S}_{a\sim b}$ raises moduli space dim, Euler char by 1, δ gives one extra H which you use to construct pairing.

δ Differential: Gluing with triangle:

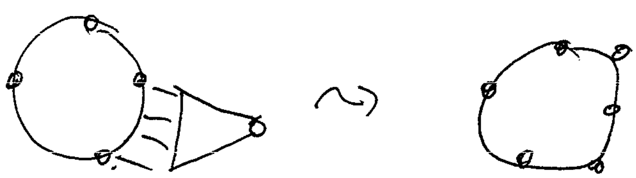


get a canonical map

$$H^{-1} \longrightarrow \mathcal{A}_\cdot(X,Y)\otimes\mathcal{A}_\cdot(Y,Z)\otimes\mathcal{A}_\cdot(Z,X)$$

$$t_\Delta,$$

(using pairing/duals).



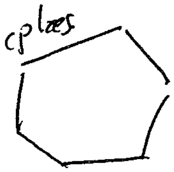
Basic fact: $\delta^2=0$, $\Delta^2=0$, $\delta\Delta+\Delta\delta=0$.

Dual construction: $CS_\cdot(\mathcal{A})$

Ex: $D = \bigcirc$ $\Rightarrow$ cyclic coh./hom. complexes

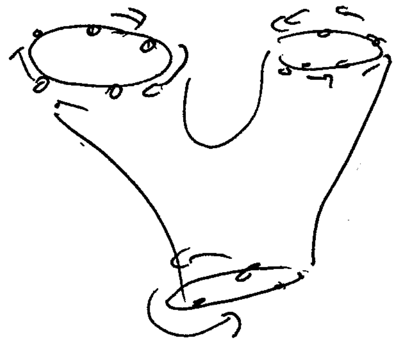
2) Hodge correlators — a map

Dolbeault cplxes [1]



$$\mathrm{Cor}_{\mathcal{H}od} := CS_\cdot(\mathcal{H}_{X/B}) \longrightarrow \mathcal{D}^{\bullet,\circ}(B)\otimes\Omega^\bullet(\mathbb{C}^2)$$

$X \to B \leftarrow B\times\mathbb{C}^2 \sim (z,w)$ twistor plane what's the map?



$\rightsquigarrow$ diff forms on

$$B \times \mathbb{C}^2$$

C^∞ here (maybe only current) — hol. here

$\boxed{\text{DGCom}}$ (so can multiply)

Preparation: $(\mathcal{A}^{\bullet,\bullet}, \partial, \bar{\partial})$ – a bigraded bicomplex

$$\begin{tikzcd} S^{m+1}(\mathcal{A}^{\bullet}[-1]) \arrow[r, "\xi_{m+1}"] \arrow[dr, "\eta_{m+1}"'] \arrow[ddr, "* = \xi+\eta"'] & \mathcal{A}^{\bullet}\otimes\Omega^1_{\mathbb{C}^2} \\ & \mathcal{A}^{\bullet}\otimes\Omega^0_{\mathbb{C}^2} \\ & \mathcal{A}^{\bullet}\otimes\Omega^{\bullet}(\mathbb{C}^2) \end{tikzcd}$$

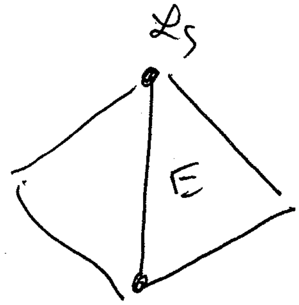
total complex

$$\xi_{m+1}(f_0,\dots,f_m,z,w) := \mathrm{Sym}_{\{0,1,\dots,m\}}\Big(\sum_{k=0}^{\infty}(-1)^k\, \frac{z\,dw - w\,dz}{2}\, f_0 \cdot z\,\partial f_1 \cdots z\,\partial f_k \cdot w\,\bar{\partial} f_{k+1}\cdots w\,\bar{\partial} f_m\Big),$$

$$\eta_{m+1}(f_0,\dots,f_m,z,w) = \mathrm{Sym}_{\{0,\dots,m\}}\Big(\sum_{k=0}^{m-1}(-1)^k\, z\,\partial f_0 \cdots z\,\partial f_k \cdot w\,\bar{\partial} f_{k+1}\cdots w\,\bar{\partial} f_m\Big)$$

$\underline{\text{Now}}$ $\mathcal{C}at = \mathcal{H}ar_{X/B}$

Green $\underline{\text{class}}$



$$G_E \in \mathcal{D}(\mathcal{L}_s^\times \otimes \mathcal{L}_t) \otimes \mathcal{D}^{0,0}(\mathcal{L}_t^\times \otimes \mathcal{L}_s) \otimes_{\mathcal{H}} \mathcal{L}_t^{1,1}$$

$$D\bar{D}\, G_E = \delta_\Delta - P_{Har} \leftarrow \text{projector to Harmonic forms}$$

Donaldson + Hitchin ⊕ $\underline{\text{Simpson}}$:

∃ harmonic metrics on $\mathcal{L}$'s ⟹ ∃ Hodge theory package

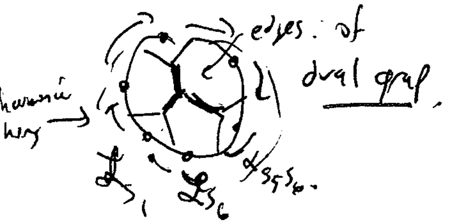
e.g. $D, \bar{D}$ (usual $\partial, \bar{\partial}$ for triv. bdles),

& $D\bar{D}$-lemma

can write: $\mathcal{A}^\cdot(\mathcal{L}_s, \mathcal{L}_t) = \mathcal{A}^{\cdot}_{Har}(-,-) \oplus \mathcal{A}_0(\mathcal{L}_s, \mathcal{L}_t)$
$\circlearrowleft P_{Har}$.

get projector, so can find G_E.

$\underline{\text{Construction}}$: Take an ideal triangulation T of S.

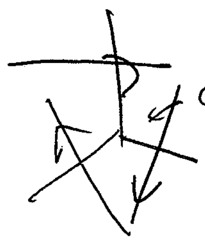


$\mathcal{X}_{/B}$ {internal edges of T} push forms to eq.

$$\mathcal{H}(G_{E_1} \circ \overset{+1}{E_1} \circ \cdots \circ G_{E_n} \cdot \overset{+1}{E_n}) \overset{+(}{\alpha}_{s_1 s_2} \wedge \cdots \wedge \alpha_{s_n} \circ \cdot \alpha_1 \wedge \text{---}$$

Get something in

$$\in CS^{\cdot}(\mathcal{H}a_{X/B}), \quad \text{differential form.}$$



corners cancel out here.

Separate volume part / edge part, get left with

$$\frac{1}{\mathrm{Aut}(T)} \sum_{\substack{\text{iso. classes}\\ \text{of ideal}\\ \text{triangulations of } T.}} \int_{\Phi} \mathrm{Tr}\, K(G_{E_1} E_1, \dots, G_{E_n} E_n)\, \alpha_1 \wedge \dots \wedge \alpha.$$

form on $B \times \mathbb{C}^2$.

Crucial fact: integral converges.

Main: Claim: correlator map

$$\mathrm{Cor} \Longleftrightarrow \underset{\text{Green class}}{\mathbb{G}} \in \overbrace{\mathcal{D}^{\cdot,q}(B) \otimes \Omega^{\cdot}(\mathbb{C}^2) \otimes CS^{\cdot}(\mathcal{H}a_{X/B})}^{\text{one big BV.}}$$

1) $\mathbb{G}$ has degree $+1$

2) Satisfies master eq'n.

$$(d + \delta)\mathbb{G} = [\mathbb{G}, \mathbb{G}] + \hbar \Delta \mathbb{G}.$$

use summation over connected $\mathcal{G}$.

If instead take

all (not nec. connected)

$$\mathcal{G}_{all} = e^{\mathcal{G}}$$

Then: $(\partial + \delta + \Delta)e^{\mathcal{G}} = 0.$

N.B.: $CS^\cdot(\mathcal{H}a_{X/B})$ is fiberwise BV,

but also a variety, so has a differential there.

$\Rightarrow$ DeRham or Dolbeault cplx. has differential

d.

$$\text{disc} \Longleftrightarrow \mathcal{G}_{Hod} \underset{\substack{A_\infty \\ \text{-quasiequiv.}}}{\hookrightarrow} D^b_{sm}(X)$$

(same for D-modules supported on smooth subvarieties w/ transverse intersections).

$g > 0 \rightsquigarrow$?? not expected by arithmetic people.