Kapranov, Generalized orders & structured surfaces

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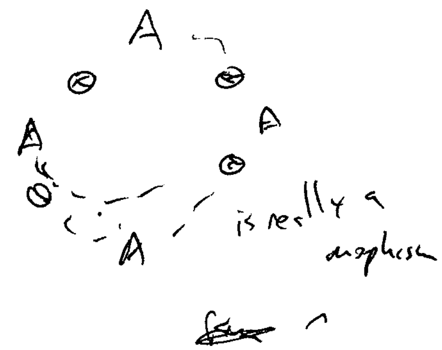
1) Two contexts of "cyclic"

Cyclic homology of algebra A

$$C_\cdot^{Hoch}(A) = (A^{\otimes n+1}, d)$$



$$C_\cdot^{cyc}(A) = (A^{\otimes n+1} / \mathbb{Z}_{n+1}^{sgn}, d)$$

Cyclic category Λ (Connes) $\quad Ob = \{[n], n = 0, 1, \ldots\}$

$\mathrm{Hom}_\Lambda([m],[n])$

$$\left(S^1, \sqrt[m+1]{1}\right) \longrightarrow \left(S^1, \sqrt[n+1]{1}\right)$$

monotone, $d = +1$.

Connes: $C^{Hoch}(A) : \Lambda^{op} \to \mathrm{Vect}$.

and $H_\cdot^{cyc}(A) := \mathrm{Tor}^{\mathrm{Fun}(\Lambda, \mathrm{Vect})}(\mathbb{C}, C^{Hoch})$.

(Rmk: But this HC^-? apparently not ..)

Important:

$\Lambda \supset \Delta$ same objects, but $\mathrm{Hom}_\Delta([m],[n])$ = monotone maps $\{0,1,\ldots,m\} \to \{0,1,\ldots,n\}$

$\mathrm{Aut}_\Lambda([n]) = \mathbb{Z}/n+1.$

$\forall\ f : [m] \xrightarrow{\in \Lambda} [n]$

$\exists!$ canonical factorization

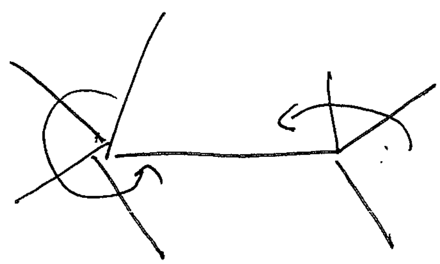
$\varphi_f \circ g_f$

$\varphi_f \in \Delta$, $g_f \in \mathbb{Z}/(n+1)$.

e.g. $\mathrm{Hom}_\Lambda([n],[0]) \cong \mathbb{Z}/(n+1)$.

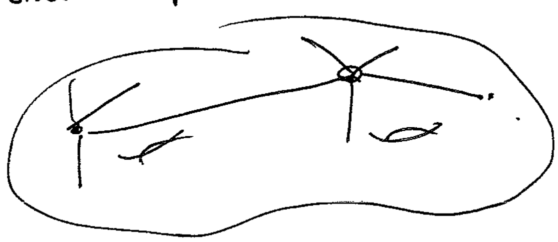
$G \circlearrowright X$, $EG \to pt$, $X \times^h_G pt$.

Ribbon graphs



cyclic order on each $Ed(v)$.

Given oriented surfaces: if Σ oriented, Γ in Σ, then Γ inherits cyclic ordering.



Γ gives cells in $\mathcal{M} = \{\mathbb{C}\text{-structures}\}$

well known, e.g. that

$$B(\text{cat of ribbon graphs}) = \coprod_{\text{top. types}} B(\text{mapping class groups})$$

② Crossed simplicial groups (Federovica - Loday)

- categories ~~which~~ like Λ, which contain Δ & have similar properties —

category: a

$$\Delta G^{w} \supset \Delta \quad \text{same objects } [n].$$

If we denote the group

$$G_n^{w} := \mathrm{Aut}_{\Delta G^{w}}([n]),$$ it's <u>required</u>: can factor

any $f : [m] \longrightarrow [n]$

$\|$ canonical fact:

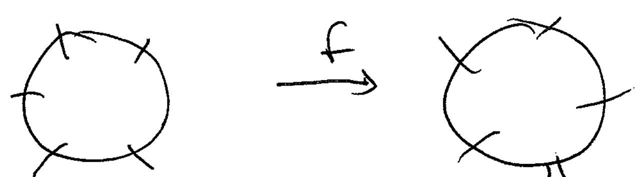
$\varphi_f \circ g_f$, $g_f \in G_m^{w}$.

<u>Examples</u>: (other than Λ and Δ)

1) Ξ dihedral category

$$G_n := D_{n+1} = \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1).$$

can be realized in terms of <u>unoriented</u> circles



unoriented $(S^1, \sqrt[n+1]{1})$

$\pm$ monotone maps $\deg f = \pm 1$

2) Paracyclic category Λ_∞

$$G_n^{w} = \mathbb{Z} \ \forall n.$$

Consider circles + universal cover

$$\mathbb{R} \longrightarrow S^1 = \mathbb{R}/\mathbb{Z}.$$

~~1-periodic~~ $\mathbb{Z}$-equivariant maps

$$(\mathbb{R}, \tfrac{1}{m+1}\mathbb{Z}) \xrightarrow{f} (\mathbb{R}, \tfrac{1}{n+1}\mathbb{Z})$$

3) N-cyclic Λ_N

N-fold cover. then $\mathcal{G}^{w}_n = \mathbb{Z}/N(n+1)$

4) N-dihedral Ξ

$$\mathcal{G}^{w}_n = D_{N(n+1)}$$

5) Quaternionic ∇

$$\mathcal{G}^{w} = Q_{n+1} = \langle \underset{\mathbb{C}}{\underset{\cap}{\sqrt[n+1]{1}}}, j \rangle \subset \mathbb{H}^* = \text{dihedral type subgroup in } SL(2)$$

$|Q_{n+1}| = 4(n+1)$

It's an extension $1 \to \mathbb{Z}/2 \to Q_{n+1} \to D_{n+1} \to 1$

6) N-~~...~~quaternionic ∇_N

$$\mathcal{G}^{w}_n = Q_{N(n+1)}.$$

Properties of these:

1) They are self-dual (not including Δ):

$$\Delta\mathcal{G}^{w} \xrightarrow{\sim} (\Delta\mathcal{G}^{w})^{op}$$
$$[n] \longmapsto [n].$$

2) $\mathcal{G}^{w}_n$ "grows linearly" with n, in this sense:

$$\overset{\overset{CF}{\|}}{\mathrm{Hom}([n],[0])} \underset{\text{duality}}{\cong} \mathrm{Hom}([0],[n]) \overset{CF}{=\!=} \mathcal{G}^{w}_0 \times \{0,1,\dots,n\}.$$

(sensible even for paracyclic).

③ <u>Structure groups in 2d</u> : connective covering

$$\mathrm{Homeo}^G(S^1) \longrightarrow \mathrm{Homeo}(S^1)$$

$$F \dashrightarrow G \xrightarrow{p} O(2) \quad \text{s.t. } p^{-1}(SO(2)) \text{ connected.}$$

$O(2) \hookrightarrow GL(2,\mathbb{R})$ h.eq.

(2-d'l version of spin)

1:1 $G^{\square}$

$\mathrm{Conf}_2 = \mathbb{Z}/2 \ltimes \mathbb{C}^\times$

(classification is purely htpy theoretic)

Klein surfaces — G^{ℓ}_{conf}

class. of all those the <u>same</u>

$G^{\square}$ - structured C^∞ surfaces

moduli spaces of conformal structure

Full list well known:

$\mathbb{R} = \widetilde{SO}(2)$ ~~SO(2)~~

$\downarrow$

$\mathrm{Spin}_N(2) = \widetilde{SO(2)}_N$

$\downarrow$

$SO(2) \hookrightarrow O(2) = \mathbb{Z}/2 \ltimes S^1$

$\tilde{O}(2) = \mathbb{Z}/2 \ltimes \mathbb{R}$

$\downarrow$

$\mathrm{Pin}_N^+(2) = \mathbb{Z}/2 \ltimes \tilde{S}^1_N$

$\downarrow$

$O(2)$

$\mathrm{Pin}_2^-(2) = Q = S^1 \sqcup jS^1 \subset \mathbb{H}$

$\mathrm{Pin}_{2N}^-(2) \to O(2)$

$M = \{X, K_X^{\otimes \frac{1}{n}}\}$

moduli spaces.

Each has a moduli space

(doesn't have univ. cover in this graph!

Match exactly. list of groups.

Provide combinatorial models.

$$\begin{tikzcd}
\Lambda_\infty \arrow[r, hook] & \Xi_\infty \arrow[d] & \\
\Lambda_N \arrow[r, hook] & \Xi_N \arrow[d] & \nabla_N \arrow[dl] \\
\Lambda \arrow[r, hook] & \Xi &
\end{tikzcd}$$

Claim: For every type of cat., $\exists$ an analogue of a ribbon graph, w/ some analogue of cyclic order that classifies the associated $\mathcal{M}$.

(4) ΔG^n-orders

$$\rho : \Delta G^{wn} \longrightarrow \mathrm{Set}$$

$$[n] \longmapsto \{0, 1, \dots, n\}$$

$$\| \, CF$$

$$\mathrm{Hom}_{\Delta G^w}([0],[n]) / G_0^w$$

$\Downarrow$

Can speak about "abstract" finite sets I, $|I| = n+1$, with ΔG^w-structure ("order").

$\| \, \mathrm{def}$

a G_n^w-torsor F. (objects are set w/ torsor action)

$\downarrow \rho \qquad \downarrow$

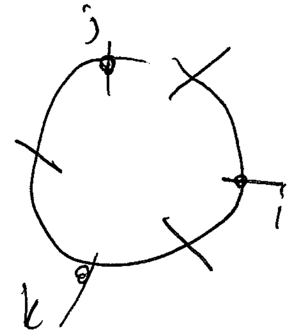
$S_{n+1} \qquad \mathrm{Isom}(\{0,1,\dots,n\})/$

Get a category $\mathcal{G} \xrightarrow{\rho} \mathrm{Sets}$

$\supseteq$

ΔG^m but more flexible.

Examples: 1) A cyclic order on I is a ternary relation.

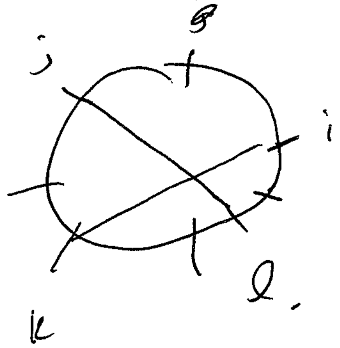


$\lambda \subseteq I^3 = \{ (i,j,k) \text{ in "right cyclic order"} \}$

can say what the conditions, ~~axs~~ axioms, etc.

2) A dihedral order is a 4-ary relation $\xi \subseteq I^4$.

no 3-pt. invts, but



right "dihedral order" if diagonals meet, for the 4 pts.

But for $|I| = 2$, 3 more morphisms

A Ξ -structure is a $\mathbb{Z}/2$ -torsor.

Sub-example: $\overline{M}_{0,I}(\mathbb{R})$ real locus

curves genus 0 w/ I marked pts.

stratified into cells.

open cells — Stasheff polytopes

labeled by dihedral orders on I, b/c

have to arrange pts. on $\mathbb{RP}^1$.

③ $\mathfrak{g}$-structured graphs Γ.

$I(\Gamma)$ = incidence category Ob = Vert $\sqcup$ Edges

structure

$$\begin{tikzcd} & I(\Gamma) \arrow[d, "F"] \arrow[dl, dashed] \\ \mathfrak{g} \arrow[r, "p"] & Set \end{tikzcd}$$

v e Morphism: $v \longrightarrow e$

inclusion of a $\frac{1}{2}$ edge.

$v \xrightarrow{F}$ $\ast$, $\mathrm{Ed}(v)$.

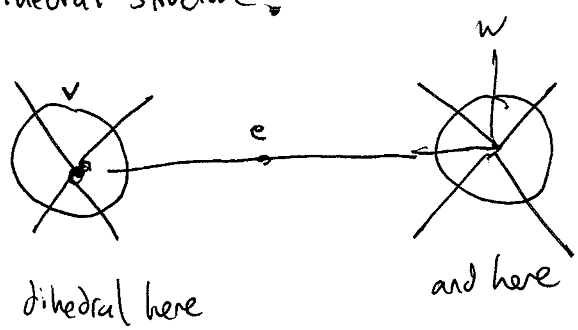
$e \xrightarrow{F}$ $\circ \longleftrightarrow \circ$ ends

A $\mathfrak{g}$-structure is a lift of F to $\mathfrak{g}$:

should be both given a $\Delta\mathfrak{G}^{\sim}$-order.

<u>Examples</u>: $\Lambda \rightsquigarrow$ Ribbon graphs.

Ξ dihedral structure:



Equivalent to Möbius graphs [Mulase, C. Brown] "monetifolds"?

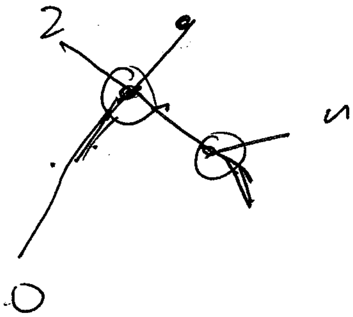
+ identif. of orientation torsors.

$$O(\mathrm{Ed}(v)) \leftarrow O(e) \longrightarrow O(\mathrm{Ed}(w))$$

$\Gamma \subseteq$ surface (oriented or not) has this structure

N.B. Naive dihedral graphs (nothing on edges) are not the same.

Cells in $\overline{M}_{0,n}(\mathbb{R}) \longleftrightarrow$ naive dihedral trees



$\Big\uparrow$ $\simeq$ heq.

B(category of such trees)

"Teichmüller spaces of unoriented surfaces" = {conform. structures} is a cell.

$\Downarrow$

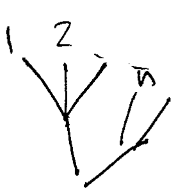
cf [S. Natanzon, M. Seppala]

$B(\{\text{Möbius graphs}\}) \simeq \coprod B$(unoriented mapping class groups).

More generally:

1) $B(\mathcal{G}$-structure graphs$) \longleftrightarrow \coprod B(G^{*}$-mapping class groups$)$.

2) Operad of $\mathcal{G}$-structure trees $\rightsquigarrow$ describes assoc. algebras w/ twisted G_0^{m}-structure.



$G_0^{m} \downarrow \mathbb{Z}/2$.

(some elts. act by homomorphisms, some by anti-homomorphisms).

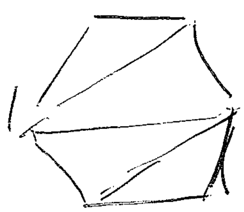
⑥ "Invariants" of $G^{\boxed{\omega}}$-structured surfaces.

numerical.	cohomological	Categorical
Frob. algebras with G_3^{ω}-action. $\longleftarrow$	CY-algebras and categories	2-Segal objects. will explain this now:

Say have

$$X : \Delta \longrightarrow \mathcal{C} \quad \text{simpl. object, } \& \quad \exists \sim \text{ in } \mathcal{C}$$

X is called 2-Segal, if:
for every polygon



$$n \quad X_n \xrightarrow[\sim]{f_T} X_T = \Big(\prod_{\substack{\sigma \in T_2 \\ \text{inner edges.}}} X_2 \Big)_{X_1,\ \forall \tau \in T_2},$$

$\forall n, T$ ← triangles

Th: If $\Delta G^{\omega} \longleftrightarrow G$ and X extends to ΔG^{ω} (and is 2-Segal), then $\forall$ ~~stable~~ smooth, e.g. hyperbolic marked $G^{\boxed{\omega}}$-structured surface S,

we can form

$$X_S = X_T \quad \forall \text{ triang. } T, \text{ ind. of } T$$

$\circlearrowright$ G-mapping class group.

$\underline{\text{Ex}}$: $\mathcal{C} = \mathbb{Z}/_{2N}$ - periodic dg-categories

$\sim$ is Morita equivalence.

$G = Spin_N(\mathbb{C})$. $\quad \mathcal{A} \in \mathcal{C}$ fixed pre-triangulated dg category in $\mathcal{C}$,

then

X: X_n = "exact n-hypersimplices" (Waldhausen constr.).

i.e. A_{ij} $0 \le i < j \le n$,

$$\begin{tikzcd} A_{ij} \arrow[r] & A_{ik} \arrow[dl] \\ A_{jk} \arrow[u] & \end{tikzcd}$$

has N-cyclic str.

$\Longrightarrow$

X_S defined for oriented N-spin surfaces.

e.g. if $\mathcal{A} = C^{(2N)}(\mathrm{Vect}_k)$,

then $X_S \sim$ top. version of Fukaya category of S-markings.

(2N-graded

$\updownarrow$

N-spin structures).