KONTSEVICH I: Fukaya Categories of Coefficients

$X \quad X, Y$ sm. projective $/\mathbb{C}$ $\qquad$ B-model

$\downarrow f \quad$ want to calculate $H^\cdot(X) = R\Gamma(X, \mathbb{C})$ in terms of structures on $Y$

$Y \qquad$ How? $R\Gamma(X) = R\Gamma(Y, Rf_* \mathbb{C}_X)$

$\qquad\qquad\quad \| \qquad\qquad\qquad\qquad \uparrow$

$\qquad\qquad H^\cdot(X) \qquad\qquad\quad$ constr. sheaf.

$$H^\cdot(X) = HP_\cdot(\cancel{\mathrm{Perf}(X)}\ D^b(Coh(X)))$$

$$=$$

$$\mathrm{Perf}(X) = \text{“}R\Gamma\text{''}(\text{coherent sheaves of } A_\infty \text{ categories on } Y).$$

Q: what can we do in the A-model??

Assume now that $X$ is Calabi-Yau, $\omega_X$ kähler form

$\mathrm{vol}_X$ hol. volume element

$\mathcal{F}(X)$ Fukaya category of $X$ $\mathbb{Z}$-graded C-Y category over Novikov field

Morally, $H^\cdot(X) = HP(\mathcal{F}(X))$.

For $y \in Y - \mathrm{Disc}(f)$ fiber $f^{-1}(y)$ is a smooth projective C-Y of smaller dimension, $n-m$.

(strictly speaking, no canonical $\mathrm{vol}_{f^{-1}(y)}$. Need to choose vol elt. on base ...)

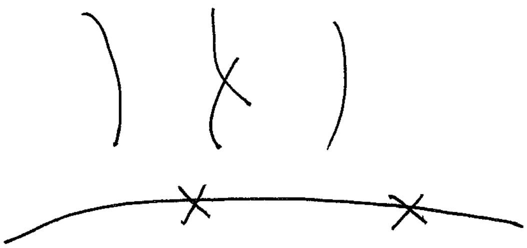
$\mathcal{F}(f^{-1}(y)$, use $\omega_X$ restricted to fiber)

Get a local system of triangulated categories on $Y - \mathrm{Disc}(f)$.

(Looks like approx. of constr sheaves, not coherent sheaves ...)

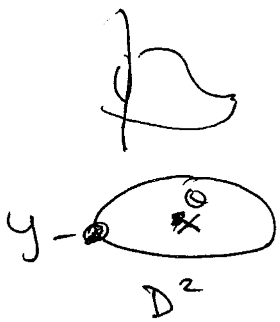
Q: Can we define a "constr sheaf" of categories?

Understand case when $\dim Y = 0$.



Consider a nhood of some point in discriminant $y_0 \in \mathrm{Disc}$

$\rightsquigarrow$




$\longrightarrow$ $= \mathcal{FS}_y$

$\mathcal{FS}$ category associated to fibration.

form a local system of non-CY categories over $S^1$ of tangent directions at $y_0$.

Moral ~~Definition~~: Pick a point $p$ in $D^2$, consider $L$ w/ boundary projecting

$\mathrm{int}\, L \subset f^{-1}(\mathrm{int}\, \mathrm{Disc})$

$\& \ \partial L \subset f^{-1}(y_-)$.

$\Rightarrow$ Also get a local system of $HP^\cdot(\mathcal{FS})$ on $S^1$.

these are = "Vanishing cycles" for $Rf_*(\mathbb{C}_X)$ $\neq$ $\nexists$ (have nearby cycles as well...)

(what to put at $y_0$ for sheaf not clear...)


In terms of $A_\infty$ categories, $\exists$ satisfactory ~~definition~~ description of situation

(Can assume fibers are not compact, open C-Y. Good enough...)

$\mathcal{FS}$ is pre-compact CY category (w/ Y. Vlassopoulos, S. Ganatra)

Assume $\mathcal{FS} = \mathrm{Perf}(A)$

$A = A_\infty$ alg., $\dim(A) < \infty$. — vanishing cycles

Say $\mathcal{F}$(generic fibre) $= \mathrm{Perf}(B)$, $\dim(B) < \infty$. — nearby cycles

$B$ is finite-dimensional C-Y algebra, meaning: (dim fibre.)

$\dim B < \infty$, and $( , ) : \mathrm{Sym}^2 B \to \mathbb{C}[-d]$ non-deg.

$(m_k(x_k, \dots, x_1), x_0)$ $\mathbb{Z}/k+1\mathbb{Z}$ invariant.

~~(w. Y. Soibelman)~~:

Thm: (Ch., K. + Y. Soibelman).

C-Y str on f.d. $A_\infty$ alg. $B \iff$ class $\in (HC(B))^*[d]$ ~~deg~~

non-deg. in following sense:

Say $m_1 = 0$, get: $\underbrace{B \to HH(B) \to HC(B) \to}_{\mathrm{tr}} \mathbb{C}$

s.t. $(b_1, b_2) \mapsto \mathrm{tr}(b_1 b_2)$ symm. bilinear is non-degenerate.

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Thm: (w. Y. Vlassopoulos):

A pre-CY str on $A$, $\dim A < \infty$

$\iff (A, \tilde{A}, \tilde{\alpha} \in CC(\tilde{A})^*, \alpha \in CC^\cdot(A)^*, \varphi : A \to \tilde{A}$

$\tilde{A}$: should be considered as "double of A"

$\varphi$: $A_\infty$ morphism

$\tilde{A}$ is a fin.-dim'l alg. ~~where $\alpha$~~
$\varphi$ $A_\infty$ morphism,

$\tilde{\alpha}$ closed s.t. it gives C-Y str on $\tilde{A}$.

and $d\alpha = \varphi^*(\tilde{\alpha})$

$\alpha$ kills pull-back of pairing,

+ non-degeneracy condition:

$$\mathrm{Total}\Big( A \xrightarrow[\varphi_1]{} \widetilde{A} \underset{CY}{\approx} \widetilde{A}^* \longrightarrow A^* \Big)$$

is quasi-iso. to 0.

composition homotopic to 0

i.e. the im $(A)$ is Lag'n subspace of $\widetilde{A}$, and on the whole have non-deg pairing. (but map might have kernel; may gives non-deg. pairing on kernel).

Roughly, $\widetilde{A}$ is $A \oplus A^*[1-d]$ with some differential)

__Ex__: $Z$ = smooth proj. Fano variety, + $s \in \Gamma(Z, K_Z^{-1})$ (which could vanish somewhere).

$A$ = End (generators of $\mathrm{Perf}(Z)$))

$\Rightarrow A$ has a pre-CY str.

$$\widetilde{A} = \mathrm{Tot}\left( \begin{array}{c} K_Z^{-1}[\pm 1] \\ \downarrow \\ Z \end{array} \right)$$

has homological vector field given by s that "squares to 0."

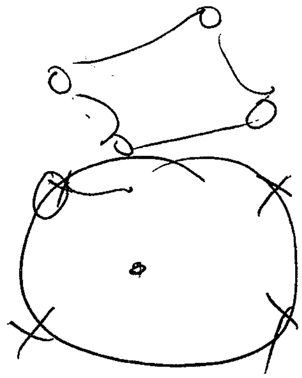
pull back generating objects to this ↑ gives real C-Y str./ dg. algebra.

s gives hom. vecfield, & consider dg scheme

if section's smooth, get C-Y hypersurface. Then, if

Further situation:
Situation II: Have a map $\widetilde{A} \to B$ what is a morphism of C-Y algebras.

Say have



many boundary pts

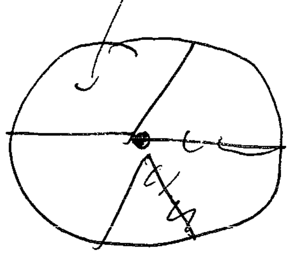
look at $L$ w/ boundary all over this

For any finite cyclically ordered set, get a generalization of $FS_{(S)}$

For any $(n)$ get $FS_{(n)}$ w/ $\mathbb{Z}/n\mathbb{Z}$ action.

<u>Suggestion:</u>

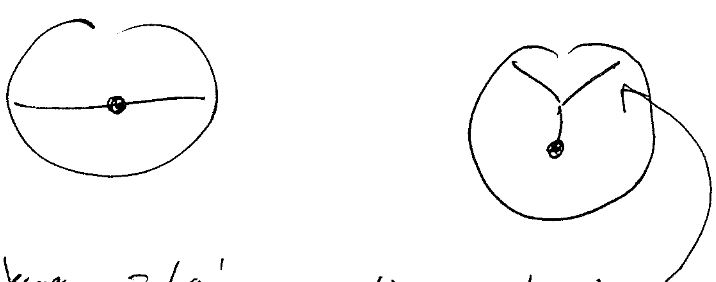
shrink $L$ s.t. Lag'n pt. to



Now, replace w/

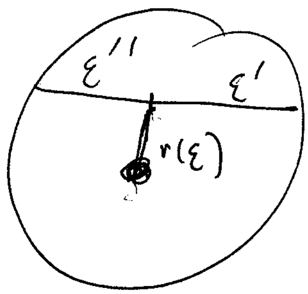
what?

$FS_{(2)}^{\mathrm{G}}$



obj categories = Lag'n manifolds proper. to
Restrict to smaller disc, get subset of $FS$: (thicken to make smooth)

Get graph w/ triple pt



$$FS \xrightarrow{r} \mathcal{F}$$

map to general fiber. $L \rightsquigarrow \partial L$

This category is: $\mathcal{E} \in FS_{(\pi)}$

w/ exact triangle

$$r(\mathcal{E}) \to \mathcal{E}' \to \mathcal{E}''$$

in $\mathcal{F}$(fiber) [?] exact triangle.

$\mathcal{E}'$, $\mathcal{E}''$ belong to Fukaya category of fibers.

If deform, get exact triangles.

Depends on how one moves things/resolves, but should give same (Kapranov's category w/ three each correspond to triangles between each)

description ...

For higher dimensional base, can make transversal section & reduce to this picture.

These seem to need C-Y structures... it would be good to be able to do this w/o the C-Y structures.

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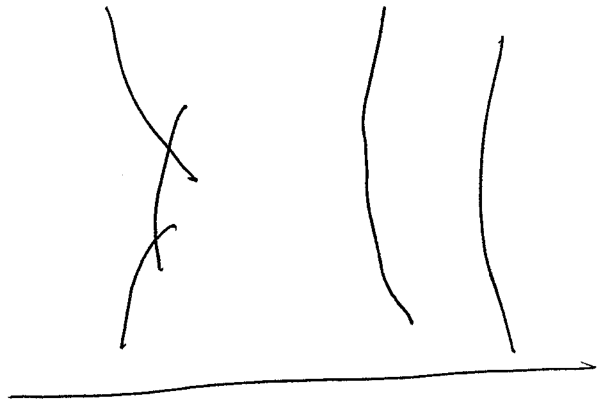
Constructible cosheaf of $A_\infty$ categories:

$$\begin{array}{l} X \quad CY \\ \downarrow f \\ Y \quad \text{Fano.} \end{array}$$

$Y \supseteq \text{Disc}(f)$ singular hypersurface



Put a category at special fibre:

quotient of $\mathcal{F}$ (smooth fibre) / vanishing cycles generated

(pretty horrible; have infinite dimensional homs)

"kill some spherical object" results in (usually complicated) objects finite dim'l in each degree, infinite

Now, get cosheaf, b/c have a map from general fibre → special fibre.

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Recall:

$X$ C-Y ⇝ stability condition.

So, slogan: "Constr. sheaf of categories w/ stability condition"

⇝ global category w/ stability condition.

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<u>Proposal</u>: (final? def'n of Fukaya category): ← not wrapped, f.d. homs.

$Y$ sympl. mfold, say $c_1 = 0$ so $\mathbb{Z}$-graded.

Lag'n skeleton ... consider all possible singular cpct. Lag'n objects (e.g.)

$L \subset Y$. e.g.

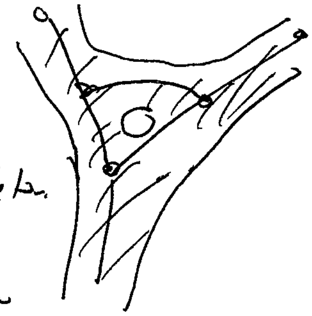
ribbon graph.

Thm good-description: symplectically small nbhd of Weinstein is

open symp. str

$\omega = d\lambda$

~~Reeb~~ & Liouville v.f. contracts to skeleton.



$\rightsquigarrow$ can make $\infty$-area objects, make flow complete

$\rightsquigarrow$ well defined Fukaya category over $\mathbb{Q}$ (no Novikov parameters necessary). combinatorial object.

when we add hol. discs, get solution of M-C eq'n depending on small parameter

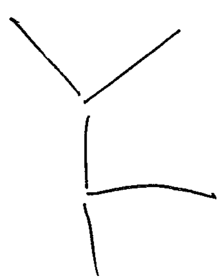
(there's also a wrapped Fukaya of Weinstein mfld.

$\rightsquigarrow \operatorname{Perf}(A_L)$ $\quad A_L$ infinite rank algebra of finite type in sense of Tsen-Vaique / $\mathbb{Q}$.

(free alg. w/ fin. many generators, differential)

Define $\mathcal{F}uk :=$ finite dimensional modules over $A_L$.

eg. Ribbon graphs in surface $\rightsquigarrow$ fin. d. complex on each edge, vertex = exact triangle.



Hol. discs gives deformation of $A_L$ depending on small parameters / ~~Novikov~~ rings.

$\mathbb{C}[[q^{\mathbb{R} \to \infty}]]$

Nov
Novikov ring, not field "small parameters"
& ~~get Novikov field~~ $\mathcal{O}_{Nov} \subset$ ring
& Novikov field

(classical limit is actual constructible sheaves)

& look at fin. dim'l modules.

Get notions of Fuk supported on any singular Lag'n,

& go to inductive limit.

(they embed into others)

N.B. on the nose, triangulated category (inductive limit of triangulated category).

Claim should be ideal thing.

<u>Hope</u>: ~~FYF~~ $\varinjlim$ ( ~~sol~~ ~~deformation~~ objects / $\mathcal{O}_{Nov}$ )

Sing. Lag'ns

$L \subseteq Y$

$\otimes$ Nov field.

$\mathcal{O}_{Nov}$

If we are given a cx-valued closed differential form / $\mathbb{C}$

$vol \in \Omega^m(Y \otimes \mathbb{C})$ $2m = \dim Y$

sufficiently close to hol. volume form,

then we should automatically get a stability condition on $F(Y)$

("mean curvature flow, try to minimize area") Stable ~~obj~~ object is $L$ s.t. $vol|_L = $ const.

More general patterns: [illegible] sheaf of $A_\infty$ cat. on $Y$. $\mathcal{F}_y$, $y \in Y$.

Assume $Y$ Kähler, get

stability condition $K_0(\mathcal{F}_y) \longrightarrow \Lambda^n(T^*_{hol}) = K_Y|_{(y)}$

$\overset{?}{\rightsquigarrow}$ Then, should get stability condition on global Fuk

$X \supset Y$

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array}$$

$$K_0(\mathcal{F}_Y) \longrightarrow \underset{\text{middle}}{H_{mid}(f^{-1}(y), \mathbb{Z})}$$

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Integrate volume fn, get vol. form on base.

At bad points, just take quotient at singular points..

& global Fukaya category

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Say $Y$ curve, $\dim Y = 1$, $\supseteq$ finite subset disc.

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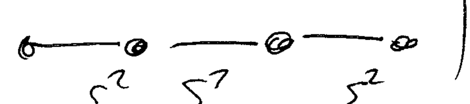
Write locally constant sheaf of categories,

Get $\pi_1(Y - \mathrm{Disc}) \longrightarrow \mathrm{Braid}_N = \mathrm{Aut}$ (2-dim'l C-Y cat.
generated by $N$-spherical object

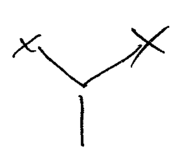
Then monodromy = reflection around spherical object




($A$ Dynkin diagram)

$\rightsquigarrow$ global 3-D C-Y category by RH?.

How? Draw graphs!



more complicated 3-d C-Y category.

stability condition ⇝ spectral curve of 3-D Hitchin system

-- spectral curves of Gaiotto-Neitzke.

Mean curvature flow --

(If you use solutions of M-C by mean curvature, get negative powers of variables, use Bridgeland asymptotics to decompose ... ).

($L_1 \subset L_2$, get embedding of objects).

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Auroux, HMS for hypersurfaces in $(\mathbb{C}^*)^n$ (w/ Abouzaid) in progress

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Geometric setup: (Abouzaid - A - Katzarkov) (see also Clarke, G-K-R, ...)

$$H = \{ f_t = \sum_{\alpha \in A} c_\alpha t^{\rho(\alpha)} x^\alpha = 0 \} \subset (\mathbb{C}^*)^n$$

$A \subseteq \mathbb{Z}^n$ $\quad x_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ (family) of hypersurfaces.

$\rho : A \to \mathbb{R}$, $t$ formal $|t| \ll 1$.

⇝ Construct SYZ mirror to $H$, $(Y, W)$ $Y$ toric CY $(n+1)$-folds, moment polytope $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R} \mid \eta \geq \varphi(\xi_1, \dots, \xi_n)\}$.

$\varphi = \operatorname{Trop} F = \max_{\alpha \in A} (\langle \alpha, \xi \rangle - \rho(\alpha))$.