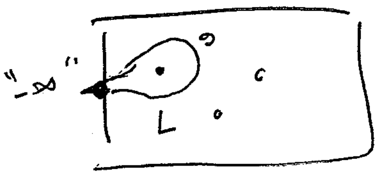
Kontsevich + Soibelman III

$$\begin{array}{l} Y \\ \downarrow W \\ \mathbb{C} \end{array} \rightsquigarrow FS \text{ category}$$



$\{z_i \in \mathbb{C}\}$ crit. values

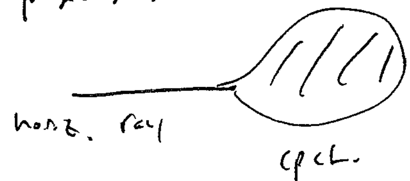
Look at $L \subseteq Y$ s.t. ∂L above "$-\infty$"
i.e. $W(\partial L) \subset \{-\infty\}$
(& transverse intersection).

$$\hom_{FS}(L_1, L_2) = \hom(L_1, L_2')$$

Main pt: not a Calabi-Yau category.

Semi-orthogonal decomposition:

∞-version: assume L projects to

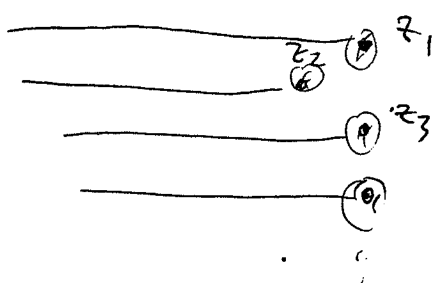


$= W(L)$

(size of square plays no role in category).

Assume $\operatorname{Im} z_i \neq \operatorname{Im} z_j \;\; \forall\, i \neq j$.

Then enumerate z_i by decreasing order



$FS_i \subset FS$ full subcat of objects projecting to ray + small disc around z_i.

N.B. For $\xi_i \in FS_i$, $\xi_j \in FS_j$, $i < j$, $\Rightarrow \hom(\xi_i, \xi_j) = 0$.

$FS := FS(Y, W)$

$\forall s \in \mathbb{C}^*$, get a family of categories over $\mathbb{C}^*$

$$FS_s = FS(Y, sW)$$

Monodromy is Serre functor for FS.

(can be seen directly: hom in opp. direction is like rotating.)

Also, each FS_i has their own Serre functor.

<u>Algebraic description</u>: local system / $\mathbb{C} \setminus \{z_i\}$

of CY categories $\mathcal{F}$ (Fukaya category of fiber)

Monodromy functor.

& each local

$$FS_i \longrightarrow \mathcal{F} \quad \text{gives a spherical functor}$$

Too much information; $\mathcal{F}$ may be very complicated!

Say • # crit. pts $= 1$; then have: — • $FS_1 = FS$, cat. w/ Serre functor

• # crit. pts. $= 2$; then have:

$$\begin{array}{l} \bullet\ FS_1 \\ \text{functor} \downarrow F_{12} \\ \bullet\ FS_2 \end{array}$$

(from pairing).

satisfying

$$\mathrm{Hom}_{FS}(\mathcal{E}_2 \in FS_2, \mathcal{E}_1 \in FS_1)$$

$$\parallel$$

$$\mathrm{Hom}_{FS_2}(\mathcal{E}_2, F_{12}(\mathcal{E}_1))$$

representing functor.

Can rotate $180°$: replace the role of FS_1, FS_2.

Replaces F_{12} by adjoint (left/right?)

$$FS_2 \xrightarrow{F_{12}^*} FS_1.$$

Q. Can you see F_{12} geometrically by a flow? A: maybe ~

Do this again; get double adjoint = conjugate by Serre functor

$$F_{12}^{**} = \mathrm{Serre}_{FS_2} \circ F_{12} \circ \mathrm{Serre}_{FS_1}$$

- #crit pts $= 3$: get

$$\begin{tikzcd}
FS_1 \arrow[d, "F_{12}"] \arrow[dd, bend right, "F_{13}"'] \\
FS_2 \arrow[d, "F_{23}"] \\
\bullet
\end{tikzcd}$$

+ natural transformation

$$F_{12} \circ F_{23} \to F_{13}$$

- 4 categories:

$$\begin{tikzcd}
FS_1 \arrow[d] \arrow[dd, bend right] \arrow[ddd, bend right] \\
FS_2 \arrow[d] \arrow[dd, bend right] \\
FS_3 \arrow[d] \\
FS_4
\end{tikzcd}$$

$\cancel{FS_{12} \circ FS_{23} \circ}$

$$F_{12} \circ F_{23} \circ F_{34} \;\Downarrow\Downarrow\; F_{14}$$

two nat transformations

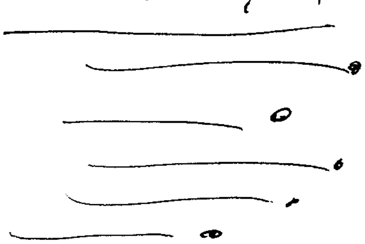
+ homotopy between natural transformations...

(same as "upper triangular A_∞ categories")

Assuming $FS_i = \mathrm{Perf}(A_i)$, the data is:

M_{ij}: bimodules $i<j$, & maps between bimodules satisfying some M-C equation.

Pretty clear, but very hairy picture! e.g. mirror of $\mathbb{P}^n$.



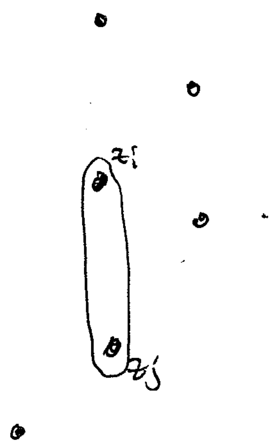
· Many high dim'l spaces.

(each FS_i will be complexes of vector spaces $= D^b(pt.)$.)

Physicists: if you have pts on a plane, & assume

no 3 ijk w/ z_i, z_j, z_k on a line;

i.e. $\mathrm{Arg}(z_i - z_j) \neq \mathrm{Arg}(z_j - z_k)$

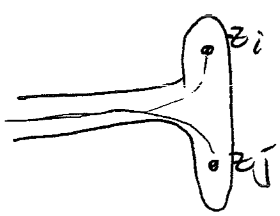


Rotating: get different bimodules.

there's one thing independent of rotation;

N_{ij}: see how local FS categories interact.

N_{ij}:



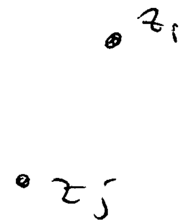
Much smaller bimodule, ~~extension~~ (e.g., for mirror of $\mathbb{P}^n$ N_{ij} all rank 1!).

Assume now that all crit. pts. are Morse

e.g. in local coords, $W = \sum_i^N x_i^2$.

Then, $FS_i = D^b(pt.)$. Then, $N_{ij} = ?$ $\theta = \theta_{ij}$.

$N_{ij} = N_{ji}^*$.



$e^{\sqrt{-1}\theta} z_i \overset{\ell}{\bullet\!\!-\!\!-\!\!-\!\!\bullet} e^{\sqrt{-1}\theta} z_j$.

by angle mult. so they lie on a line

$\theta = \mathrm{Arg}(z_j - z_i)$.

Now, consider gradient flow for $\mathrm{Re}(e^{i\theta} W)$

$=$ Ham. flow for $\mathrm{Im}(e^{i\theta} W)$.

So, all trajectories project to horizontal line.

so $W^{-1}(\bullet \xleftarrow{\ell} \bullet)$ $2n-1$ dim'l manifold

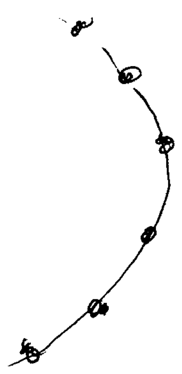
On this mfold, have two crit. values, & attracting/repelling grad. lines

Quotient by flow: $2n-2$ dim'l,
should get finitely ~~many~~ gradient
lines connecting these things,
have Maslov index --
will get basis of N_{ij} := gradient flow lines
(really, taking HF(vanishing cycles in generic fibre).

Goal: Describe FS in terms of N_{ij}
+ tensors satisfying M-C equation.

M_{ij} should be expressable via N_{ij}?

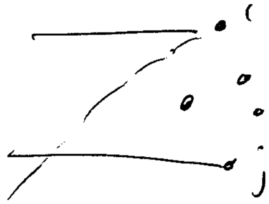
Nice situation:



pts. on convex curve to right

Then, $N_{ij} = M_{ij}$.

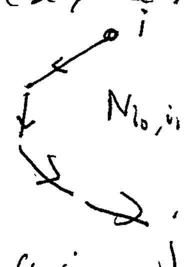
In general, to calculate homs



have to move past pts, get a spectral sequence:

$i<j \quad M_{ij} \Longleftarrow \bigoplus$ $N_{i_0,i_1} \otimes N_{i_1,i_2} \otimes \cdots \otimes N_{i_{k-1},i_k}$

spectral seq.



w/ shifts/gradings.

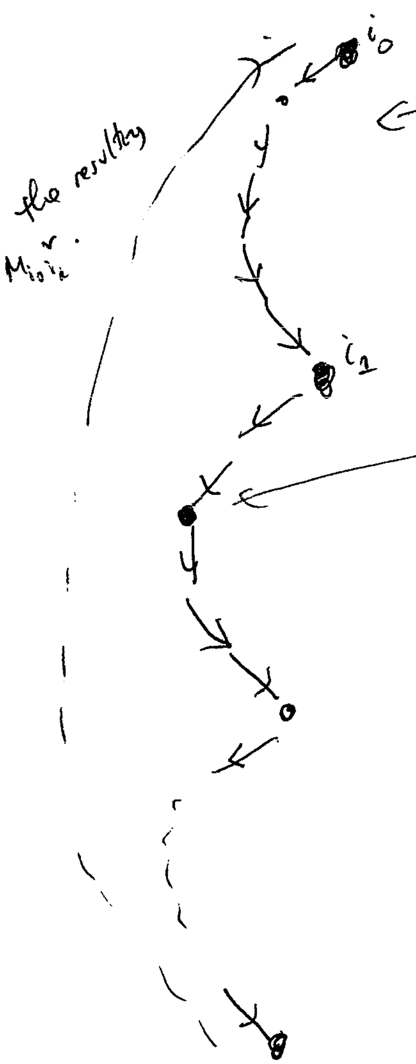
convex to left opposite

Cohomological Hochschild complex of this upper triangular algebra is:

for any upper triangular algebra:

$$\begin{pmatrix} \mathbb{C} & & M_{ij} \\ & \ddots & \\ 0 & & \mathbb{C} \end{pmatrix} \quad \text{is} \quad \prod_{i_0 < \cdots < i_k} \mathrm{Hom}\left(M_{i_0 i_1} \otimes \cdots \otimes M_{i_{k-1} i_k},\ M_{i_0, i_k}\right).$$

Apply to this situation: get: describe $M_{i_0 i_2}$ as

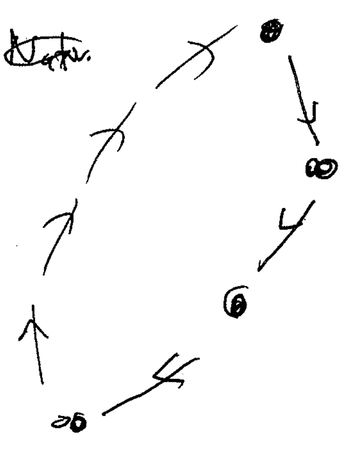


Claim: same pictures appear 2^n times.

e.g. can remove this point, still admissible

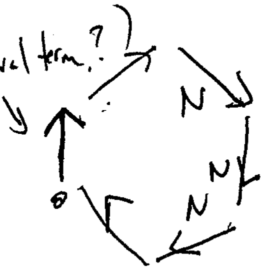
So ~~the~~ these terms cancel in spectral sequence, except for ~~2~~ appearances or remaining surviving terms convex to right.

~~Notn.~~



Situation becomes completely symmetric

Claim: Cohomological Hochschild complex of ~~$\mathcal{A}$~~ $\mathcal{FS}$

$\simeq$ (q-iso) $\bigoplus C($  $)$ of $\bigotimes N_{i_k, i_{k+1}}$.

all convex polygons

dual term?

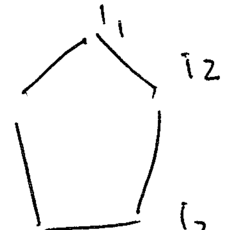
what's the differential?. want to say it's an L_∞ alg,

& M-C eq'n to solve ...

Solution of M-C equation should be;

$\left(\alpha_{P_0} \in \bigotimes_{j \in \mathbb{Z}/k} N_{i_j i_{j+1}} \right)$

P polygon.

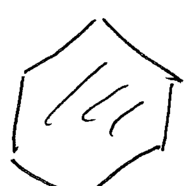


Something involving J-curves w/ ∂ on thimbles, but why non-toric contributions don't contribute is shocking.

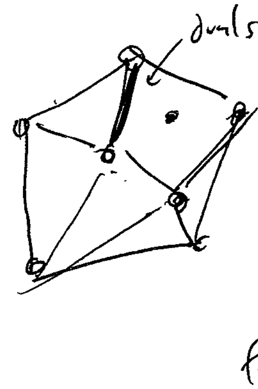
Claim: this is the L_∞ structure! (gives)

Don't know what the M-C elt. should be...

(~~all ways to~~ some decompositions)

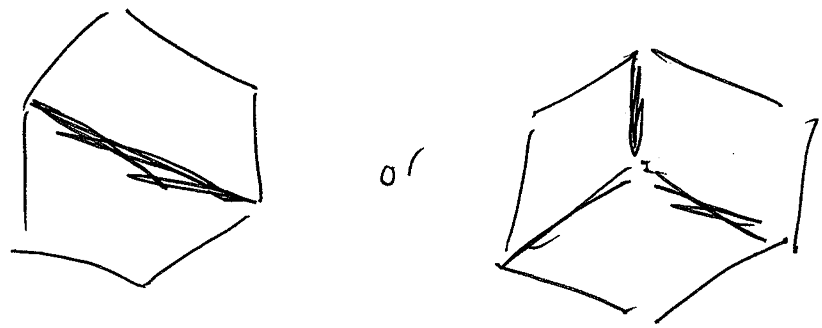
MC eq'n $\partial \left(\alpha \right.$  $\left. \right)$ = polynomial in α_P's, ← smaller polygons.

dual space; can cancel.



say $P = \bigcup P_{\text{faces}}$.

$\bigotimes_{\text{faces}} \alpha_{\text{faces}} \longrightarrow$ larger tensor product what admissible decompositions? taut decomposition



taut: ① # inner edges = 2 # faces − 3

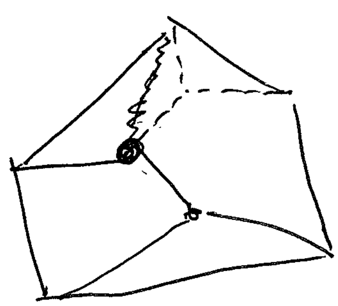
② ∃ convex function ~~~~ on P w/ ~~linearity~~ domain of linearity given by this function.

(there are some signs).

$d^2=0$ point: you can decompose a double taut decomposition in exactly two ways

e.g.

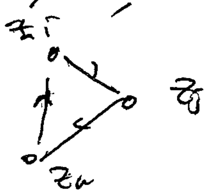
1) —— vs. and ----
2) —|— vs. ~~~~



& from this, one can go back to A_∞ structures on FS categories.

what information are we using? The A_{ij}'s, + data of, for each triangle, whether triangle oriented clockwise or ccw

⇝ "higher Bruhat order."



Gaiotto-Moore-Witten give simple different description of A_∞ structure.

also [illegible] 2010

(Kapranov–Soibelman ...)

non-Morse-Smale-flows ...)?

in terms of infrared limit

(idea:

"webs"

Jhol. ⟶ deformation using $t\frac{\partial W}{\partial \phi}$

lim $t \to \infty$ is "infrared"!

(cf. also Fan–Jarvis–Ruan).

N_{ij}

N_{jl}

M_{ij}