Pascaleff, Equivariant Lag'n Branes & Representations

joint w/ Y. Lekili.

Spaces: G $\mathbb{C}$ - simple alg. group e.g. $G = SL_n(\mathbb{C})$

B Borel subgroup e.g. $\begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix}$ $\nabla < SL_n$.

Borel-Weil: G/B has line bundles $\mathcal{L}_\lambda$ s.t. $H^0(\mathcal{L}_\lambda) = V_\lambda$ irrep w/ highest weight λ.

λ dominant weight for G

e.g. $SL_2 \rightsquigarrow \mathbb{P}^1$ $SL_n \rightsquigarrow Fl_n$

Mirror (Rietsch): $\mathcal{R} \subset G^L/B^L$, W Lie theoretically defined superpot'l.

Langlands dual gp.

complement of a pair of opposite Schubert divisors. (Flag & opp. Flag)

E.g. SL_2 $\mathbb{C}^* \subset \mathbb{P}^1$ e.g. Schubert stratification  $W = z + \frac{1}{z}$

SL_3 $\mathcal{R} \subset Fl_3$

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \begin{matrix} z \neq 0 \\ xy - z \neq 0 \end{matrix} \right\} \qquad W = x + y + \frac{x}{z} + \frac{y}{xy - z}$$

HMS: $\mathcal{L}_\lambda \longleftrightarrow \overset{\text{some}}{L_\lambda} \subset \mathcal{R}$

$$V_\lambda = \Gamma(\mathcal{L}_\lambda) = HF^0(L_0, L_\lambda).$$

- interesting feature of symplectic side: canonical basis of intersection pts, coming from intersection pts. of Lagrangians (need $\psi' = 0$)

The group action is hidden on the A-side.

G does not naturally act on $\mathcal{R}$.

Equivariant structures let us see this action in terms of symplectic geometry (holomorphic curves).

~~Geom~~ Parallel in rep. theory: Geometric Satake correspondence:

$$Gr_{G^L} = G^L(K)/G(\mathcal{O}) \qquad K = \mathbb{C}((t)) \qquad \mathcal{O} = \mathbb{C}[[t]]$$

G

$\circlearrowleft$ G rep.

$$V_\lambda \longleftrightarrow IH^*\left(\overline{G^L(\mathcal{O})[\lambda]}\right)$$

& M-V basis gives basis of this rep'n

Tannakian formalism tells us that these are rep. of some group.

Q: what acts on Floer cohomology? Ans: $SH^*(R)$

Lie algebra (need to work infinitesimally)

$$\mathfrak{g} \subseteq \overset{\text{vector fields}}{Vect(G/B)} \subset H^0(\wedge^\bullet T_{G/B})$$

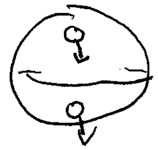
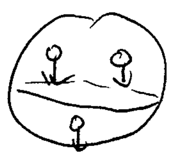
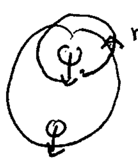
$$\overset{HKR}{=} HH^*(Coh(G/B))$$

$$\downarrow$$

$$SH^*(R) = HH^*(W(R)) = HH^*(Coh((G/B) \setminus \overset{\text{anticanonical}}{D}))$$

Symplectic cohomology: Floer cohomology for "convex" sympl. manifolds (e.g. Stein m'folds or affine varieties)

- generated by periodic orbits of a "quadratic" Hamiltonian* $SC^*(R)$
- differential & operations are defined by Riemann surfaces.

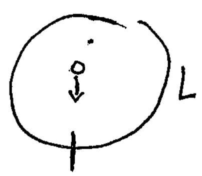
d:  product  BV  allowed to rotate.

$SH^*(R)$ is a BV algebra: • graded commutative product

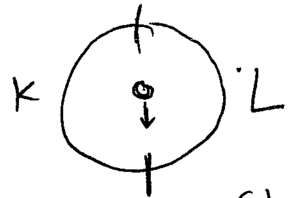
• Lie bracket of degree -1.

in particular: $SH^1(R)$ is a Lie algebra. (typically ∞-dim'l).

<u>Action</u>: on wrapped Floer cohomology, via closed-open string maps.



gives a map $\phi_1^0 : SC^1(R) \to CW^1(L,L)$.



$\phi_1^1 : SC^1(R) \to \mathrm{Hom}(CW^*(K,L), CW^*(K,L))$

fits together into a map $SC^*(R) \to \mathcal{CC}^*(\mathcal{W}(R))$.

Also need to consider maps with multiple $SC^*(R)$ input.
(N. Sheridan).

<u>Equivariant structure</u>: choose $c_L : SC^1(R) \to CW^0(L,L)$

s.t. $dc_L = \phi_1^0$

makes it <u>invariant</u>.

ϕ_1^0 : Vect $\to \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$

(K, c_K), (L, c_L)

Action of $z \in SC^1(R)$: $\phi_1^1(z) - \mu^2(c_L(z), \cdot) + \mu^2(\cdot, c_K(z))$

This is a well-defined map $SH^1(R) \to \mathrm{End}(HW^*(K,L))$

(Building on Seidel–Solomon).

More compatibility conditions to make this a map of Lie algebras.

Since SH^1 is typically ∞-dim'l, we're looking for a sub-algebra

$$\mathfrak{g} \subset SH^1(R)$$

$$SH^1(R) = \mathrm{Vect}\left((G/B) \setminus D \right) \supseteq \mathrm{Vect}(G/B).$$

(should be distinguished using the superpotential W).

Observations: For any R:

1) $H^1(R) \longrightarrow SH^1(R)$ its image is an abelian subalgebra, (maybe not maximal??) (Cartan)

2) extra $H_1(R; \mathbb{Z})$ - grading. (maybe root lattice ?? dual of ~~Cartan~~, (root lattice).

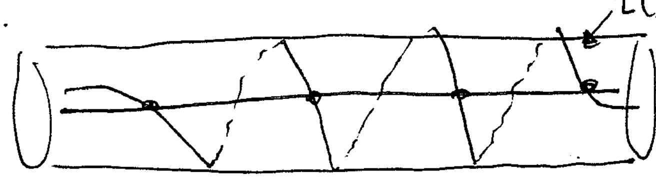
3) If K and L are simply connected Lag'ns, then $HW^*(K,L)$ carries a relative $H_1(R;\mathbb{Z})$ grading. (grading set is a torsor over))

4) In general, we expect to find $\mathfrak{g} = \mathrm{Lie}\, G$ inside $SH^1(R) \subseteq G^\vee/B^\vee)$ in a way compatible w/ 1) - 3).

Case of $G = SL_2(\mathbb{C})$: $R = \mathbb{C}^\times$, $W = \cancel{z} + \cancel{\frac{1}{z}}$. $L(n)$ $\forall n$.

$G/B = \mathbb{P}^1$.

$L(3)$



$L(0)$

$$SH^0(R) = \mathbb{K}[z, z^{-1}]$$

$$SH^1(R) = \mathbb{K}[z, z^{-1}]\partial_z = \mathrm{Vect}(\mathbb{G}_m), \quad H^1(R).$$

$$\cup$$

$$\langle \partial_z, z\partial_z, z^2\partial_z \rangle = \mathfrak{sl}_2 = \mathfrak{g}$$

Thm: $L(k)$ may be made equivariant $c_{L(k)} \equiv \emptyset$.

(when embedded this way)

then the subspace

$$HF^0(L(0), L(k)) \subset HW^0(L(0), L(k))$$

is $\mathfrak{g}$-stable and is $\mathfrak{g}$-equivariantly isomorphic to $\Gamma(\mathbb{P}^1, \mathcal{O}(k))$

canonical SL_2-equiv. structure.