Ruddat, Conifold Transitions, Tropical Geometry, & Mirror Symmetry

Agenda:

- Mirror symmetry for conifolds can be realized in the Gross–Siebert program
- Morrison's conjecture
- Tropical/affine cohomology

Def: A conifold is a CY3 with at most A_1-singularities ($xy = zw$)

Let $\bar{X}$ be a conifold.

$\exists$ two ways to obtain smooth CY3 from $\bar{X}$.

$Bl_{x=0}\bar{X} = Bl_{y=0}\bar{X}$ $Y_1 \leftarrow \bar{X} \rightsquigarrow X_\varepsilon$ $xy - zw = \varepsilon$

$Y_1 \updownarrow$ flop $\bar{X} \rightsquigarrow X_\varepsilon$ smoothly

$Bl_{w=0}\bar{X} = Bl_{z=0}\bar{X}$ $Y_2 \rightsquigarrow X_\varepsilon$ conifold transition.

There are generally obstructions, global.

$$\mathbb{P}^1 = S^2 \leftrightsquigarrow p \rightsquigarrow S^3.$$

Reid's web conjecture:

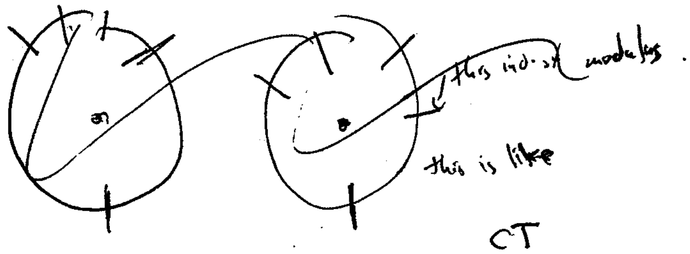
All C-Y 3-manifolds are connected by sequences of conifold transitions (+ reverse)

Conjecture: (Morrison) For (X, Y) C-Ys w/ $X^\vee, Y^\vee$ C-Y mirrors:

$$\begin{array}{ccc} Y & \overset{CT}{\rightsquigarrow} & X \\ \text{MS}\updownarrow & & \updownarrow\text{MS} \\ Y^\vee & \underset{CT}{\overset{\text{reverse!}}{\leftsquigarrow}} & X^\vee \end{array}$$

Note: for MS, you need a maximal degeneration.

$$\begin{array}{ccccc} Y_t & \leftarrow & \bar{X}_t & \rightarrow & X_t \\ & & \updownarrow & & \\ & & \bar{X}_0 & & \\ & & \updownarrow \text{GS} & & \\ & & \bar{X}_0^\vee & \leftrightsquigarrow & X_t \end{array}$$



$$\begin{tikzcd}
Y_t \arrow[rr, bend left, "CT"] & \overline{X}_t \arrow[l] \arrow[r] & X_t^{\varepsilon} \\
& \overline{X}_0 \arrow[u, Rightarrow, "\overline{\alpha}\ \text{log str}"'] & \\
& \check{\overline{X}}_0 \arrow[u, Leftrightarrow, "\text{Discrete Legendre trans form}"'] \arrow[ddl, bend left, "\beta"] \arrow[dd, Rightarrow, "\check{\beta}\ \text{log str.}"] & \\
& & \\
\check{Y}_t^{\varepsilon} & \check{\overline{X}}_t \arrow[l, squiggly] \arrow[r, squiggly] \arrow[l, bend left=60, "CT'"] & \check{X}_t
\end{tikzcd}$$

Can construct smoothings from other log structures.

As part of the theorem, need to extend G-S to conifold points

discriminant

$\Delta \subseteq B :=$ dual intersection complex of $\overline{X}_0$.

$\check{\Delta} \subseteq \check{B} :=$ dual intersection complex of $\check{X}_0$.

$i : B \setminus \Delta \hookrightarrow B \qquad i : \check{B} \setminus \check{\Delta} \hookrightarrow \check{B}$.

$\Lambda_{B\setminus\Delta} =$ Integral tangent vectors on smooth locus.

$\check{\Lambda}_{B\setminus\Delta} =$ cotangent

$S \subseteq B$ subset of 4-valent points in Δ: tropical conifold points.

SYZ, tropical cycles, tropical deformation

Locally $\exists$ two types of degenerations of manifolds

(1) $xy - tzw = 0$

planar

$\Big\updownarrow$ ms

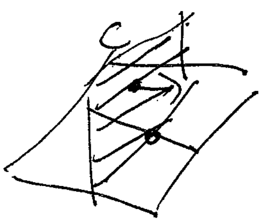
(2) $xy - tu = 0$

$zw - tu = 0$

spatial

Thm: (Castro-Bernard, Matessi);

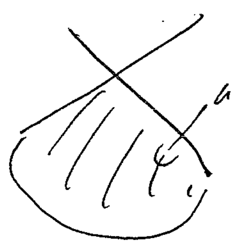
Let $C \subset \check{B}$ be a tropical 2-cycle



planar

C attaches to one of the branches

spatial



attaching to discriminant

carrying a section of Λ.

If $\exists\, C$, $\Rightarrow$ Friedman-Tian, ~~and~~ satisfied for $\check{X}_t$

Smith-Thomas-Yau for $\overline{X_t}$.

Conjecture: This is an if and only if.

How does this relate to our work?

Thm: CBM cycles naturally live in

tangent sheaf w/ decoration.

$$H_2(\check{B}, \check{\Delta}; i_* \Lambda^{\mathcal{P}})$$

$$\big\downarrow T \quad \text{natural map into}$$

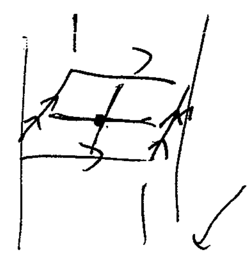
$$H^1(\check{B}\setminus\check{S}, i_*\Lambda)$$

controls smoothing.

Conjecture $\Longleftrightarrow$ T is surjective. (it's not injective.)

Counterexample:

Rem: It fails for



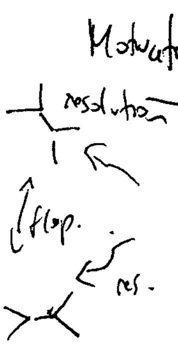
$= \check{B}$ /

real f.b. over T^2 s.t. discriminant locus has 1 conifold point (spatial conifold point).

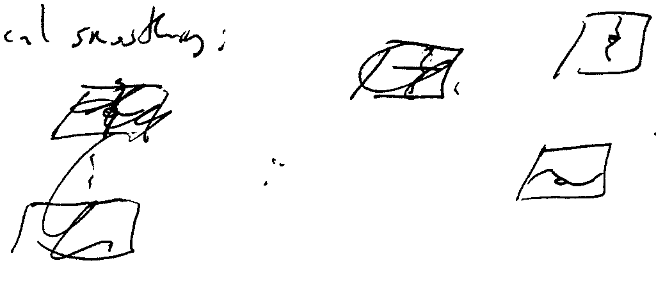
$\rightsquigarrow$ possible to smooth.

but! doesn't exist a two cycle inducing this! (cots discrete leg. transform has planar)

Motivation (CBM): Can produce a tropical smoothing:



$\rightsquigarrow$ separate:



) smoothing

(T)heorem: (j. w/ Matessi)

(image one dimension up).