Abouzaid:

X symplectic
$\pi \downarrow$ smooth Lagrangian ~~torus fibration~~ X_q.
Q

$$\{z \in \Lambda \mid |z| = 1\}$$

SYZ mirror: $Y_q = H^1(X_q, U_\Lambda) = (U_\Lambda)^n$

this has a ring of fxns. of the form

completion of Laurent polys. over $H_1(X, \mathbb{Z})$

$$\left\{ \sum_{\gamma \in H_1(X_q, \mathbb{Z})} a_\gamma z^\gamma \;\middle|\; \lim_{|\gamma| \to \infty} |a_\gamma| = \infty \right\}.$$

Rigid analytic mirror:

analytic structure $\longrightarrow$ $Y = \coprod_{q \in Q} Y_q$

tropical manifold $\Leftrightarrow$ $SL(n, \mathbb{Z})$-manifold } induced by ω on X.

$\exists$ a canonical identification of

$$T_q Q \cong H^1(X_q, \mathbb{R})$$

Letting $\Lambda^* =$ non-zero elements of Λ, can map

$$U_\Lambda \longrightarrow \Lambda^* \xrightarrow{val} \mathbb{R}.$$

Can identify a nbhd P_q of $q \in Q$ w/ a subset of $H^1(X_q, \mathbb{R})$

We obtain an embedding

$$Y_p \cong \coprod_{p \in P} \subseteq H^1(X_q; \Lambda^*) \subseteq (\Lambda^*)^{n}.$$

Analytic fns. $\mathcal{O}_p$ on Y_p are Laurent series on which converge on Y_p.

Thm: (A. '14): There exists a class $\alpha_X \in H^2_{an}(Y, \mathcal{O}^*)$ and a faithful embedding

obstructs X having a Pin Lag'n section. (having a Lag'n section $\otimes w_2$(base))!

$$\mathcal{F}uk(X) \hookrightarrow D^b_{\alpha_X}Coh(Y)$$

twisted coherent sheaves

Claim This functor is fully faithful.

Key Idea: Find an object of $\mathcal{F}(X)$ or rather in an enlargement of $\mathcal{F}(X)$ having HF^* iso. to $\mathcal{O}_{Y_q}$.

Toy case: Assume X is exact symplectic, $L \subseteq X$ exact.

Fix a basepoint $* \in L$

Let $\mathcal{P}$ be the "local system" on L w/ fiber at a point $x \in L$ $C_*(\mathcal{P}_{*x}L, \mathbb{Z})$

paths $[0,1] \to L$

$0 \to *$

$1 \to x$

$$HF^*(\mathcal{P}, \mathcal{P}) = H^*(L, Hom(\mathcal{P}, \mathcal{P})).$$

Floer

Eilenberg-Moore S.S.

$\cong H_*(\Omega_* L)$.

If L is a torus, then

$$H_0(\Omega_* \cancel{T}^n) \cong \mathbb{Z}[H_1(T^n, \mathbb{Z})].$$

$\cong$ Laurent polynomials.

B.C.

$C^0(Q)$ $\pi_1 Q$

Koszul dual

$C_* \Omega_q Q$.

If $K \subset X$ is another exact Lag'n, then

$HF^*(P, X)$ is a module over Laurent polynomials, i.e. a coherent sheaf on $(\mathbb{C}^*)^n$.

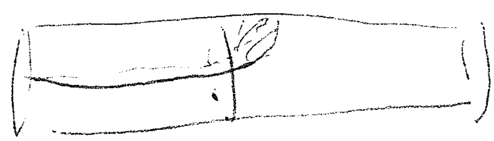
(Assume that $\pi_2(Q) = 0$).

$\Rightarrow$ Fibers X_q do not bound any discs.

Idea: Given any $K \subset X$, we get a coherent sheaf on Y_q by computing $HF^*(\hat{P}_q, K)$ where $\hat{P}_q$ is a "completion" of P_q local system on fiber X_q.

Have to complete because there may be "holomorphic strips" in infinitely many classes.

E.g.



Yields a Laurent series in Floer differential instead of polynomial.

Consequence of Gromov compactness: "$\exists$ only finitely many discs s.t. the length of the boundary along X_q is bounded." (almost).

$$C_*(\mathcal{P}_{*x} L) \qquad C_*(\hat{\mathcal{P}}_{*x} L)$$ completion w.r.t.

associate a real number to each component of path space $\mathcal{P}_{xx}$ which is minimal length of a representative.

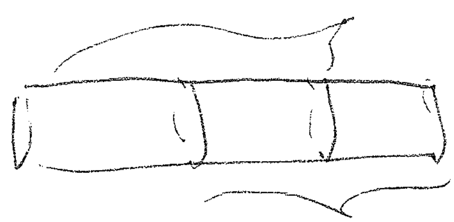
$HF^*(\hat{\mathcal{P}}_q, \hat{\mathcal{P}}_q) \cong$ isomorphic to a completion of Laurent polynomials

$\cong \mathcal{O}_q$ ← i.e. fns. on Y_q.

For $K \subset X$, we have a well defined $HF^*(\hat{\mathcal{P}}_q, K)$ a module over $\mathcal{O}_q$, is a coherent sheaf on Y_q.

"Fukaya's Found. Thm. Family Floer Theory"

<u>FFTFFT</u>: $\forall K \subset X$, $\exists P$ a nbhd of $q \in Q$ s.t. $HF^*(\hat{\mathcal{P}}_P, K)$ is a well-defined $\mathcal{O}_P$-module.



By a local-to-global argument, we get a coherent sheaf on Y.

Let $\mathcal{O}_Q$ be a "covering category* for Y."

(Pick an ordered cover of Q by polygons P_i ⇝ gives a cover Y_i of Y

<u>Ob</u> i in the index set

<u>Hom</u>: $\mathrm{Hom}(i,j) \cong \begin{cases} \mathcal{O}_{Y_i \cap Y_j} & i \leq j \\ 0 & \text{otherwise} \end{cases}$

(<u>assume</u> if $i<j$, then $Y_i \subseteq Y_j$.

<u>Pt</u>: $\mathcal{O}$-modules on Y ⟷ Modules over $\mathcal{O}_Q$.

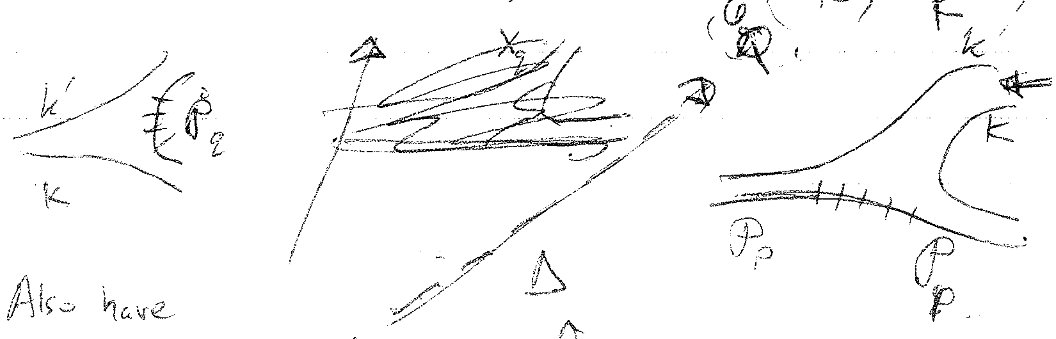
Given $K \subseteq X$, we obtain

$\mathcal{L}_K$: left $\qquad\qquad \mathcal{R}_K$ right.

given by $\mathcal{L}_K(i) \cong HF^*(\hat{P}_{p_i}, K) \qquad \mathcal{R}_K(i) \cong HF^*(K, \hat{P}_{p_i})$

Let's say our functor is $K \longrightarrow \mathcal{L}_K$.

$\rightsquigarrow$ get a map $HF^*(K, K') \xleftrightarrow{\text{know last year}} \mathrm{Hom}_{\mathcal{O}_Q}(\mathcal{L}_K, \mathcal{L}_{K'})$.



Also have

$\mathcal{R}_K \otimes_{\mathcal{O}_Q} \mathcal{L}_{K'}$ $\qquad$ module duality + multiplication.

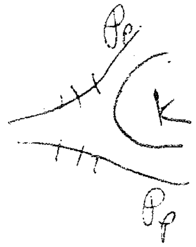
Claim: Composition is an isomorphism.

i.e. there is a natural equivalence $\quad$ diag. bimodule

$$\mathcal{R}_K \longrightarrow \mathrm{Hom}_{\mathcal{O}_Q}(\mathcal{L}_K, \Delta_{\mathcal{O}_Q})$$

i.e. $\mathcal{R}_K$ is the "smooth dual" to $\mathcal{L}_K$

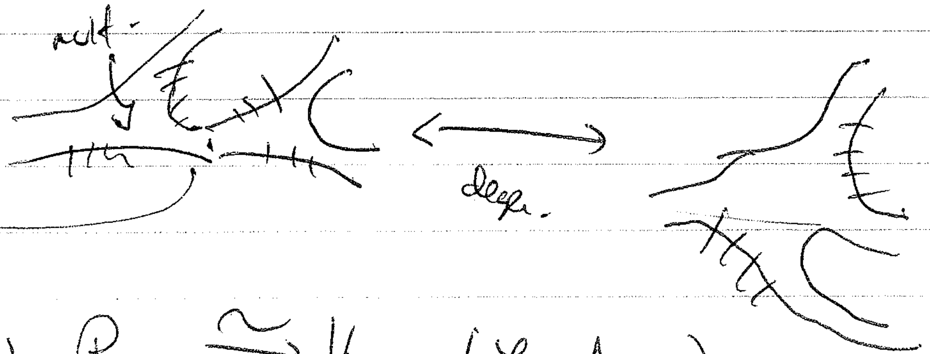
i.e. $\exists$ a map (adjoint to) $\mathcal{L}_K \otimes_K \mathcal{R}_K \longrightarrow \Delta_{\mathcal{O}_Q}$ map of bimodules.



Warning: Have to check "convergence" at every step

Need: ① Diagram counting

cobordism argument:



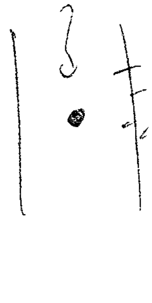
② Need $R_K \xrightarrow{\sim} \mathrm{Hom}_{QQ}(\mathcal{L}_K, \Delta_Q)$

so R_K & $\mathcal{L}_K$ are indeed dual

(uses Cardy rel'n twice;

compose & glue:

identity map on ~~[illegible]~~



$\hom(P, B) \otimes_J \mathcal{C}$

$\downarrow$

$\hom(B, B \otimes_J \mathcal{C})$