Kaledin, Cyclic homology of a different kind

in progress

A/k assoc. algebra flat / comm. ring k

$HP(A)$

prod. total cplx.

$$\begin{array}{ccccccc} \to & A & \to & A & \to & & \\ & \uparrow & & \uparrow & & & \\ \to & A^{\otimes 2} & \xrightarrow{1} & A^{\otimes 3} & \to & & \sigma = (-1)^n \cdot \text{order} \\ & \updownarrow b' & & \uparrow b & & & \text{n permutation} \\ \xrightarrow{1+\sigma+\sigma'} & A^{\otimes 3} & \xrightarrow{1-\sigma} & A^{\otimes 3} & & & \end{array}$$

If $k \supset \mathbb{Q}$, then sum-total cplx. is acyclic.

<u>Def</u>: $HP'(A)$ sum total cplx.

<u>Kont</u>: char p, may be meaningful.

It has been

$$\mathbb{Z}_p/p = A, \text{ then } H_{DR,dc}(A) = B_{DR}/p$$
$$\overline{H}_{DR,dc} = B_{cris}/p$$

<u>Beilinson - Bhatt</u> p-adic Hodge:

A comm. ring

$A_\cdot$ free simplicial res.

$\Omega^\cdot(A_\cdot)$ de Rh. bicplx.

prod. total $\swarrow$ $\searrow$ sum total

$H_{DR,dc}(A) \in \Lambda^i \Omega(A)$

if k p. torsion

Cartier iso

$H^i_{DR}(A_1) \cong \Omega^i(A)$

$\Lambda^i \Omega(A) \Rightarrow \overline{H}_{DR,dc}(A)$

Thm: (1) $HP'(A)$ is (locally) torsion

(2) If A is smooth (as a dg. algebra), we have an exact triangle

$$HP'_\bullet(A) \to HP_\bullet(A) \to HP(A) \otimes \mathbb{Q} \to$$

Example: $A = k = \mathbb{Z}$. $HP'(A) = \mathbb{Q}/\mathbb{Z}((u))[1]$
deg $u = 2$,

$$\begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Q} \arrow[ll, bend left] \end{tikzcd}$$

($\mathbb{Z}/p$; these two are the same).

Genl (non-smooth).

Prop: Assume $pk = 0$. Then, we have a spectral sequence

$$HH_\bullet(A^{(1)})((u^{-1})) \Rightarrow HP'_\bullet(A) \quad \text{similar to } HdR.$$

Frob twist)
(conjugate spectral sequence)

Laurent power series in u^{-1}. (HH bounded above, NOT $((u))$).

$\Rightarrow$ Morita invariant, etc.

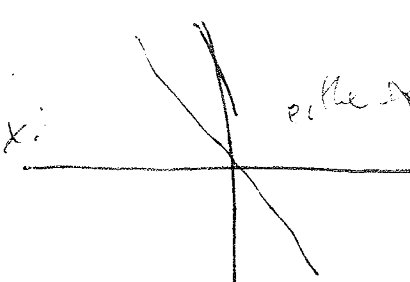
For DGA: $A^\bullet/k$ dga.

Hochschild homology: sum total

$\Rightarrow$ if dga's quasi-iso, HH. same.

(Positselsky: product-total cplx.)
we do not do.

2D cplx:



either $\oplus$ sum or $\prod$ in one direction (4 choices).

$CH_\cdot(A^\cdot)$ sum total cplx. (non-unital HH:

$B : CH_\cdot(A) \to CH_\cdot(A)[1]$

$N(i)$ the same .

$\ni \sum_i \sum_{\substack{\text{finitely } j = n(i) \\ \leq i}} a_{j,n} u^{j}$. $a_{i,n} \in A^{\otimes n}$

$\forall i$, j bdded by $N(i)$.

$CH_\cdot(A[[u]][u^{-1}])$ $\to CH_\cdot(A^\cdot((u))) \longrightarrow CH_\cdot(A^\cdot)((u))$

$d + Bu$ $\qquad d + Bu$

$CH_\cdot(A^\cdot((u^{-1}))) \to CH^\cdot(A^\cdot)((u^{-1}))$.

$\sum x_k \otimes - \otimes x_1$

each $x_i = \sum y_i^j u^j$.

$$\widetilde{HP}(A) \xrightarrow{\text{III}} \widehat{HP}'(A) \xrightarrow{\text{IV}} HP'(A)$$

$$\widetilde{HP}(A) \xrightarrow{\text{II}} \widehat{HP}_\cdot(A) \xrightarrow{\text{I}} HP_\cdot(A)$$

$k \supseteq \mathbb{Q}$ vanishes mostly

Thm: (1) If A is smooth I is an iso.

call "tempered"?

(2) A bdded above ~~str~~ / k-mod $\Rightarrow$ III iso.

(3) " " below / " " $\Rightarrow$ II iso.

(4) A bounded above / k $\Rightarrow$ IV is an iso.

& — — / $A^{op} \otimes A$.

(smooth & proper deg 0 concentrated).

All the theory defined via $A^\circ_\# \in$ Functors $(\Delta, k\text{-mod})$.

$HP_\cdot(E_\cdot)$, $\widehat{HP}_\cdot(E)$ only depend on $E_\cdot \in D(\Lambda, k)$. — derived cat. of cyclic k-mod

defined by taking cot&invariants??

so give invt.

$\overline{HP}_\cdot'(E)$, $\overline{HP}_\cdot'(\overline{E})$ are not q-iso invt

but $\overline{HP}_\cdot'(A_\cdot)$, $HP'(A_\cdot)$ are ok.

& $HP_\cdot'(A_\cdot)$ Morita invt. why?

Prop: $\exists$ SS $HH_\cdot(A^{(1)})((u^{-1})) \Rightarrow HP'(A^\cdot)$.

& $\uparrow$ is Morita invt.

u^{-2}!

About proofs:

Got $\mathbb{K} = \overline{\mathbb{F}}_p$ to get rid of Frob. twists.

V k-vect.

Tate homology

Then, $\check{H}_i(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \cong V$, $\forall i \in \mathbb{Z}$.

(*) $\varepsilon : 1+\sigma+\dots+\sigma^{p-1} : V^{\otimes p}$ ↻

$\ker \varepsilon \big/ \operatorname{coker} \varepsilon \xrightarrow{1-\varepsilon}$; $\sim V$, V cohomology.

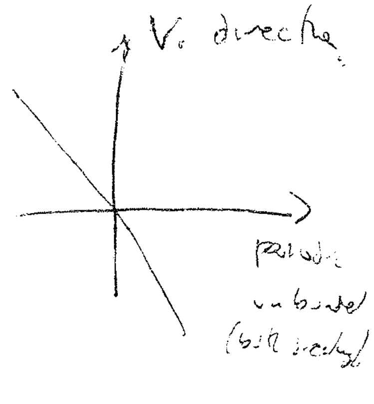
Eventually gives rise to a Cartier iso for algebras.

What do we do if V is a complex of vector spaces? $V_\cdot$

$\check{C}_\cdot(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$ is a bicomplex.

complete

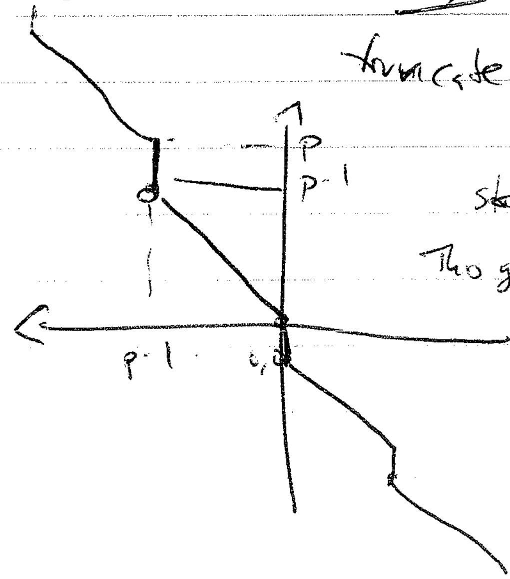
totalize in one of two ways



$\leadsto$ $\left[V_\cdot^{\otimes p} \xrightarrow{1-\sigma} V_\cdot^{\otimes p}\right]((u^{-1}))$

$d = (1-\sigma) + u\varepsilon$. (*). Treat this bicomplex as a filtered cplx. w.r.t. (stupid) filtration of vertical direction.

$\rightsquigarrow \mathcal{DF}(k)$ ← filtered vector spaces, has *many* t-structures, so can truncate in other ways:



~~stupid filt~~

This generates a canonical truncation.

$\tau^{(p)}_{\leq 0}$ (& a stupid one, above?)

^ everything below, along differential is 0.

<u>Fact</u>: $gr^{i}_{\tau(p)}\left(\check{C}^{\cdot}(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})\right) \cong V$.

$\forall i \in \mathbb{Z}$.

true for odd prime b/c $p-1/2$ works. & $p=2$ b/c cplx is 1-periodic.

for applications, need $\mathbb{Z}/p^n\mathbb{Z}$.

<u>Degeneration</u>: <u>Take cohomology</u> (some, shifts by 1):

$A^{\cdot}$ DGA:

$\tau_{\leq 0}\check{C}^{\cdot}(\mathbb{Z}/p\mathbb{Z}, A^{\cdot\otimes p})$ DG Algebra.

$\shortparallel$

$P(A^{\cdot})$ is an iterated extension of A by $A^{\cdot}[i]$ $i \in \mathbb{Z}_+$.

split lifting to $\mathbb{Z}/p^2$.

$\Updownarrow$

Ass. grade: $\begin{bmatrix} A^{\cdot} \\ A(1) \\ A(2) \end{bmatrix}$

Prop: Co-periodic spectral sequence for A degenerates if (f??) $P(A^{\cdot}) \to A$ <u>splits</u>