Kapranov:

Perverse Schobers & top. Fukaya category

w/ V. Schechtman & Y. Soibelman

categorification of ~~Perverse~~

① Perverse sheaves.

$$U \mapsto H^i_{K \cap U}(U, \mathcal{F})$$

A. Cohomology w/ support

$$\begin{tikzcd} H^i_K(X, \mathcal{F}) \arrow[rr] & & H^i(X, \mathcal{F}) \arrow[dl] \\ & H^i(X \setminus K, \mathcal{F}) \arrow[ul] & \end{tikzcd}$$

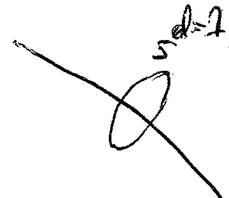
(K closed; X space, $\mathcal{F}$ sheaf)

$\underline{H}^i_K(\mathcal{F})$ sheaf on K.

Often: "sphericity": only one $\underline{H}^i_K \neq 0$.

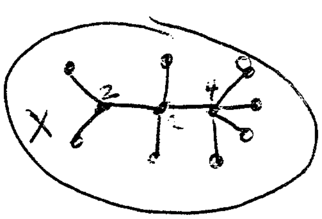
Ex: X - C^∞ mfold, $\supset K$ closed submfold codim d.

$\underline{H}^d_K(\underline{\mathbb{Z}}_X) =$ coorientation local system



S^{d-1} - sphere in transv. direction.

2) X surface $\supset K$ graph



Only $\underline{H}^1_K(\mathbb{Z}_X) \neq 0$

& $\dim(\text{stalk}_x) = \text{val}(x) - 1$ ← valency

sheaf of balanced assignments of integers

B. Perverse Sheaves

X $\mathbb{C}$-mfold, $\quad S = (X_\alpha)$ $\mathbb{C}$-stratification.
smooth, loc. closed.

$\boxed{<}$ $\quad \mathcal{F} \in D^b Sh$ (const sheaves)

called S-smooth if $\forall$ $\underline{H}^i(\mathcal{F})\big|_{\forall X_\alpha}$ loc. constant of finite rank.

S-perverse if: S-smooth and:

(1) $\underline{H}^i(\mathcal{F})$ supported on codim $\geq i$.

(2) $\mathbb{H}^i_{X_\alpha}(\mathcal{F}) = 0 \quad i < \operatorname{codim}_{\mathbb{C}} X_\alpha$.

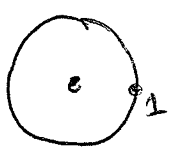
- heart of a t-structure, called $\boxed{\mathrm{Perv}(X, S)}$, abelian,
- with duality $\mathcal{F}^* := \underline{R\mathrm{Hom}}(\mathcal{F}, \underline{\mathbb{C}}_X)$.
- Noetherian, Artinian,

$\underline{\text{R-H}}$: holonomic regular $\mathcal{D}_X$-modules, smooth w.r.t. X_α.
Local objects which behave as sheaves.

Often: $\sim$ Rep(Quiver, Rels).

C. Examples

$\mathcal{D} =: X$ (disc) $\quad S = \{0, \mathcal{D} \setminus 0\}$



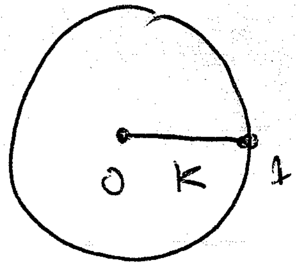
(1) $\Rightarrow \mathcal{F}|_{X_{\text{generic}}}$ is a local system

$$\mathrm{Perv}(\mathcal{D}, 0) \simeq \left\{ \Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi \;\middle|\; \begin{array}{l} \overset{\text{monodromy}}{T_1} = \mathrm{Id}_\Psi - vu \text{ is invertible} \\ (\text{fin dim.} \Rightarrow T_0 = \mathrm{Id}_\Phi - uv \text{ invert.}) \end{array} \right\}$$

Φ vanishing $\to$ not a stalk.

Ψ nearby $= \mathcal{F}_1$ stalk at 1

Explicitly:



Only $\mathbb{H}^1_K(\mathcal{F}) \neq 0$ (so, sphericity).

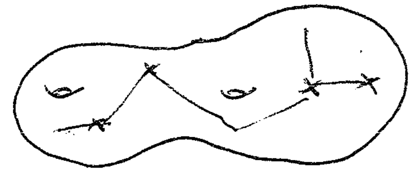
ϕ = stalk at 0

ψ = everywhere else.

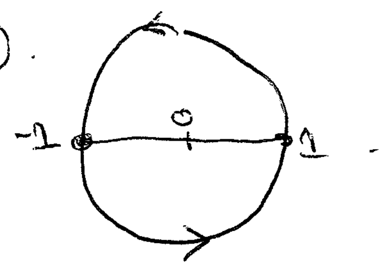
v is std. restriction, other map by duality.

2) More generally, if X is a surface, K a graph, $\mathcal{F}$ a perverse sheaf,

only $\mathbb{H}^1_K(\mathcal{F}) \neq 0$.



2') Take $K = [-1, 1] \subseteq D$.



"Dirac vs. Schrodinger desc. of perv. sheaves (taking sq. root).

$\mathcal{F} \in Perv(D, 0)$

$\mathbb{H}^i_K(\mathcal{F})$ has 3 stalks,

$$\begin{tikzcd} E_- \arrow[r, shift right, "\delta_-"'] & E_0 \arrow[l, shift right, "\gamma_-"'] \arrow[r, shift left, "\gamma_+"] & E_+ \arrow[l, shift left, "\delta_+"] \end{tikzcd} ;$$

half-monodromies

$\gamma_+ \delta_+ = \mathrm{id}$

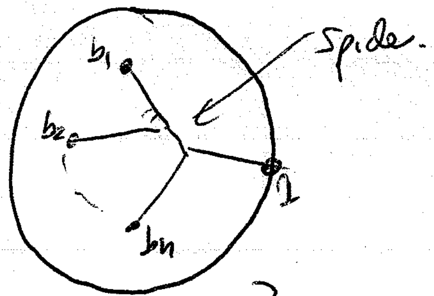
$\gamma_- \delta_- = \mathrm{Id}.$

$\gamma_+ \delta_- : E_- \to E_+$

$\gamma_- \delta_+ : E_+ \to E_-$

"half monodromies"

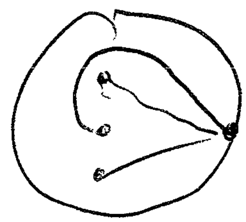
③ Perv$(\mathcal{D}, b_1, \ldots, b_n)$



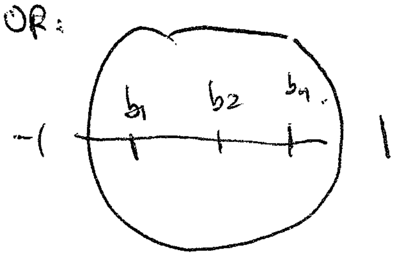
Fixing this spider get:

$$\left\{ \Psi \; \begin{array}{c} \underset{v_1}{\overset{u_1}{\rightleftarrows}} \Phi_1 \\ \vdots \\ \underset{v_n}{\overset{u_n}{\rightleftarrows}} \Phi_n \end{array} \;\middle|\; T_i = Id - v_i u_i \text{ invertible} \right\}.$$

But, $\exists$ different descriptions:



OR:

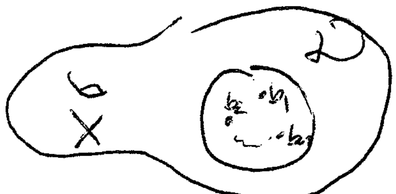


$\rightsquigarrow \quad E_{-1} \rightleftarrows F_1 \rightleftarrows E_{12} \rightleftarrows F_2 \rightleftarrows \; - -$

All descriptions are equivalent

④ X surface $\supset B = \{b_1, \ldots, b_n\}$. put inside a disk $\mathcal{D}$.

Perv$(X, B) \simeq$



$\Gamma_{X,B}$ mapping class group acts

$\{$ Perv$(\mathcal{D}, B)$ + local system on $X - \mathcal{D} \}$.

(3) Want to categorify.

$$\begin{tikzcd}
\text{ab. groups} & \text{(dg) triangulated categories} \arrow[l, "K_0"'] \\
\text{sheaves of } \arrow[u] & \text{stacks of } \arrow[u] \text{ on } X \arrow[l]
\end{tikzcd}$$

German Schobers.

Perverse sheaves $\leftarrow ?$ Perverse Schobers

"what are cplxes. of categories?"

Point: these simple diagrams can be categorified easily, ad hoc; even w/o general definition

Ex. $\mathrm{Perv}(\mathcal{D}, 0) \leftarrow\!\!-\,-$ Spherical functors (R. Anno, T. Logvinenko)

$$\mathcal{D}_0 \underset{F^*}{\overset{F}{\rightleftarrows}} \mathcal{D}_1$$ exact functor of triangulated category

right adjoint, so

$$\mathrm{Id}|_{\mathcal{D}_1} \xleftarrow{\varepsilon} FF^*$$

F called spherical if e.g. $T_1 = \mathrm{Cone}(\varepsilon)$, $T_0 = \mathrm{Cone}(e)$ are equivalences of categories (so corresp. to monodromy)

(ones categorify the diffeos.

$$F^*F \xrightarrow{e} \mathrm{Id}_{\mathcal{D}_0}$$

Ex. $E \in \mathcal{D}_1$ spherical obj: $\mathrm{Ext}^i(E,E) = \begin{cases} \mathbb{C}, & i = 0, n \\ 0 & \text{otherwise} \end{cases}$ & $\mathrm{Serre}(E) \simeq E[n]$.

Then, $D^b\mathrm{Vect} \xrightarrow{F} \mathcal{D}$

$\mathbb{C} \longmapsto E$ is a spherical functor.

for ex. X 3-fold, $\supset C \simeq \mathbb{P}^1$, s.t. $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

$\mathcal{O}_C \in D^b\mathrm{Coh}(X)$ is a spherical obj.

2) Direct description of $\mathrm{Perv}(\mathcal{D}, O)$

↑

"spherical pair"

$\mathcal{D}^{\pm}$ admissible subcats

$$\begin{tikzcd} \mathcal{D}^- \arrow[r, shift left, "\delta_-"] & \mathcal{D} \arrow[l, shift left, "\gamma_-"] \arrow[r, shift right, "\gamma_+"'] & \mathcal{D}^+ \arrow[l, shift right, "\delta_+"'] \end{tikzcd}$$

$\langle \mathcal{D}_-^{\perp}, \mathcal{D}_- \rangle$ proj. along ${}^{\perp}\mathcal{D}_-$ $\mathcal{D}_-^{\perp}$ ${}^{\perp}\mathcal{D}_+$

$\langle \mathcal{D}_-, \mathcal{D}_-^{\perp} \rangle$.

s.t. $\mathcal{D}_- \to \mathcal{D} \to \mathcal{D}_+$

$\mathcal{D}_+ \to \mathcal{D} \to \mathcal{D}_-$ equivalences

<u>For example:</u> 3fold

$X_+ \supset C$ $(-1,-1)$-curve.

$$\begin{tikzcd} & Z = Bl_p(Y) \arrow[dl, "q_-"'] \arrow[dr, "q_+"] & \\ X_- \arrow[dr] & & X_+ \arrow[dl, "\text{contract}(C)"] \\ & Y & \end{tikzcd}$$

flop

Flop-flop functors:

$$\begin{tikzcd} D^b Coh\, X_+ \arrow[r, shift left, "T_{-+}"] & D^b Coh\, X_- \arrow[l, shift left, "T_{+-}"] \end{tikzcd}$$

composition is spherical twist;

diagram gives spherical pair.

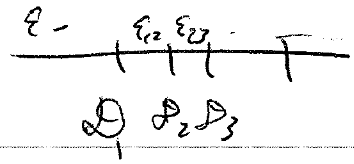
$\mathcal{D}_{\pm} = D^b Coh(X_{\pm}) \subseteq \mathcal{D} = \langle q_+^* \mathcal{D}_+, q_-^* \mathcal{D}_- \rangle$.

3) Schobers on $(\mathcal{D}, b_1, \dots, b_n)$

= diagrams of spherical functors,

$$\begin{tikzcd} & \mathcal{D}_1 \arrow[dl] \\ \mathcal{D}' & \vdots \\ & \mathcal{D}_n \arrow[ul] \end{tikzcd}$$

or of spherical pairs



4) On a surface: by gluing.

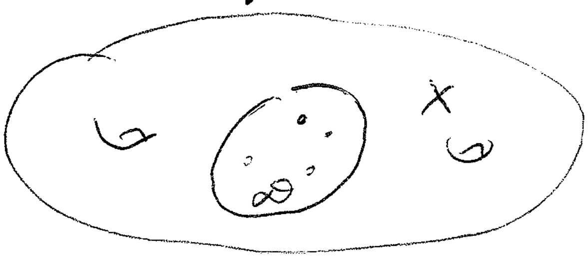


diagram of spherical functors on $\mathcal{D}$,
+ a local system of triangulated categories
on $X \setminus \mathcal{D}$, identified on the boundary.

<u>Basic example</u>: Fukaya Schober of a symplectic fibration.

$$\begin{array}{ll} (Y, \omega) & \mathcal{F}(\pi) \text{ on } X \\ \pi \big\downarrow & \Phi_{b_i} = \text{local Fukaya Seidel category} \\ X \ni b_1, \dots, b_n & \Psi = \text{Fuk(fiber)}. \end{array}$$

surface   singular values

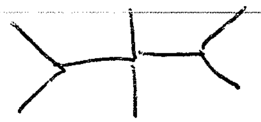
(5) Schobers as coefficients

Kontsevich proposed (earlier Miami?):

1) define ~~F(X)~~ Fukaya category of X w/ coefficients (e.g. local systems of categories).

2) For X "Stein" (exact, etc.), use a singular Lag'n skeleton $K \subseteq X$ to ~~define~~ get a natural ~~ℓ~~ sheaf of (triangulated) categories $\mathcal{A}_X$ on K so that $Fuk(X) \simeq \mathbb{H}^0(K, \mathcal{A}_X)$.

Ex: X = surface $\supset K$ graph



$(\mathcal{A}_K)_x = Rep(A_{n-1}, D^b Vect))$.
$n = val(x)$.

Two parts:

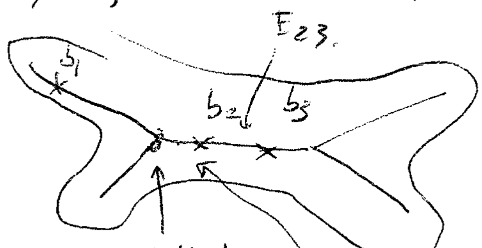
(!1): Coefficients should be perverse Schobers. Local systems are a particular case.

(!2): $K_0(\mathcal{A}_X)$ should be $\underline{H}^n_K(\mathbb{Z}_X)$. ← (so should be categorification of).

sheaf of abelian groups $\quad n = \frac{1}{2}\dim_{\mathbb{R}} X$

Combine these for surfaces X

$B = \{b_1, \ldots, b_n\} \subset X$ $\quad$ K spanning graph



stalk here is cat. of exact triangles in ambient category

E_2 ~~E_3~~ from Dirac delta fn.

For a Schober $\mathfrak{S}$ on X, we can define directly a not perverse sheaf F of categories

$\mathbb{H}^1_K(\mathfrak{S})$ on K.
$\shortparallel$
$\mathcal{A}_K(\mathfrak{S})$.

Def: $Fuk(X, \mathfrak{S}) = \mathbb{H}^0(K, \mathcal{A}_K(\mathfrak{S}))$

not how to define

Independence of K; similarly to Dyckerhoff-K.
(combinatorial Teichmüller space (contractible)).

Is it dependent on choice of skeleton? Same as ~~K~~ Dyckerhoff-K on no coeffs; unique in a canonical way.

"Fuk is a categorification of middle dim'l coh. given by Lag'n cycles"