Keating: Symplectic properties of ADE Milnor fibres.

$f: \mathbb{C}^{n+1}: \mathcal{U} \to \mathbb{C}$, holo., $df|_0 = 0$, $df \neq 0$ on $B_r^k(0)$

WLOG $f(0) = 0$.

Germ of f is an isolated hypersurface singularity.

(Today: Mostly $n=2$):

$B_\delta(0) \cap f^{-1}(0)$: singular.

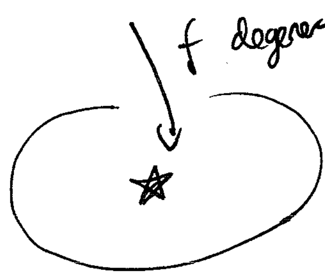
But, $B_\delta(0) \cap f^{-1}(\varepsilon_\delta)$: smooth $\leftarrow$ Milnor fibre of f, M_f.

exact symplectic; w/ contact type boundary.

Symplectic Picard-Lefschetz theory:

$\mathbb{C}^{n+1}$

f degenerate



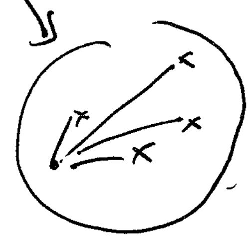
$\mathbb{C}$

$\overset{\text{deform } f}{\rightsquigarrow}$ to $\tilde{f}$, ex. Morse.

$\mathbb{C}^{n+1}$

$\tilde{f}$

crit. pts. $=$ Milnor # $= \mu$.



Morse crit pts.

– v. paths $\rightsquigarrow$ v. cycles $\subset f^{-1}(\varepsilon_\delta)$ Lag'n sphere.

Milnor: $M_f \simeq \bigvee_\mu S^n$

non-intersecting paths $\to$ distinguished collection of van. cycles.

distinguished collection gives basis for $H_n(M_f)$.

• Intersection form $\langle , \rangle$ or $\cdot$ ($n=2$: $L \cdot L = -\chi(L)$).

$\tau_L \in \pi_0(\mathrm{Symp}^c M_f)$; Dehn twists.

ADE case = modality-zero = simple = $\langle , \rangle$ negative definite

e.g. A_n $x^{n+1} + y^2 + z^2$ •—•—• - - •

Positive modality singularities:

e.g. $T_{p,q,r}: x^p + y^q + z^r + xyz$.

• $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$.

• $a \in \mathbb{C}^*$ Hly. many values.

$\langle , \rangle$ semi-def. or indef.

ADE case:

<u>Ritter</u>: Only exact (cpct) Lag'ns are spheres.

Positive modality

<u>Thm</u>: $\exists$ exact $S^1 \times S^{n-1}$, primitive in homology, & Maslov class 0.

(*).

$$\underline{\text{Pf. strategy}}: \quad \overset{\swarrow \text{gen. of singularity}}{[f]} \longrightarrow [g] \quad f + \varepsilon p \sim g. \quad M_g \underset{\substack{\text{exact} \\ \text{symplectic}}}{\hookrightarrow} M_f$$

"f adjacent to g."

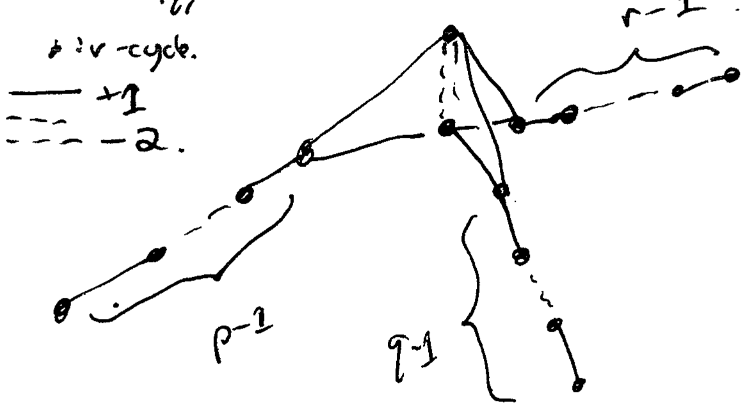
enough to show (*) for $T_{p,q,r}$.

$(p,q,r) = (3,3,3), (4,4,2), (6,3,2)$ &

$a = 0$.

<u>Rest</u>: 3-variable $T_{p,q,r}$. & $\mathcal{T}_{p,q,r} = M_{T_{p,q,r}}$.

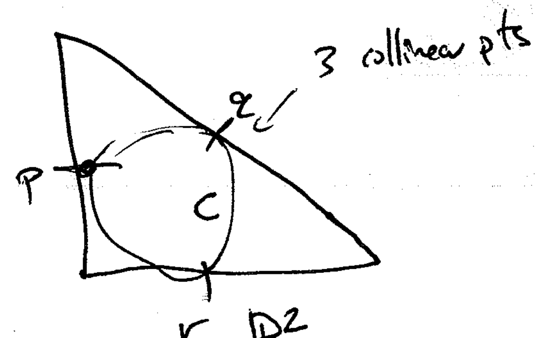
$T_{p,q,r}$ (Gabrielov):



Geometrically, get a description of $\mathcal{C}_{p,q,r}$. w/ exactly these intersections (no cancellations).

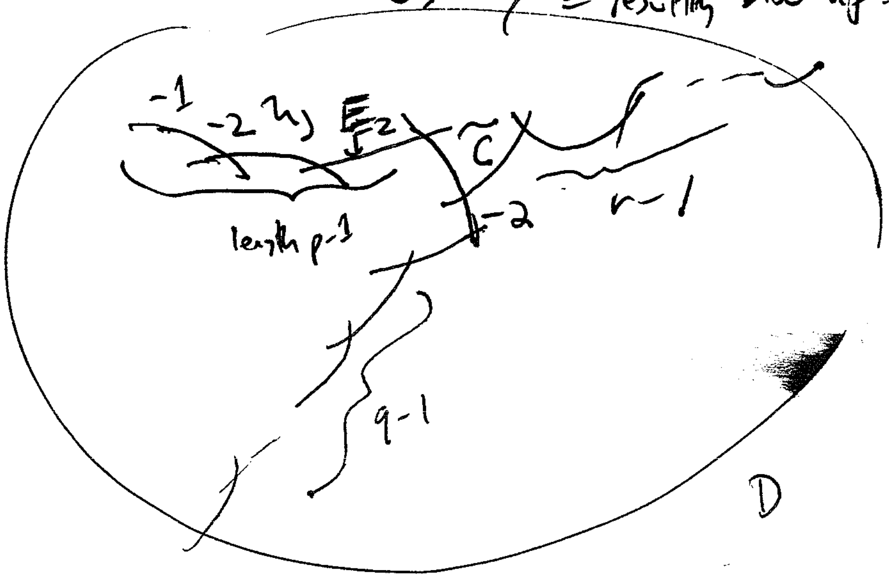
($\mathcal{T}_{p,q,r}$ described as total space of Lefschetz fibration.)

$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$: Gross–Hacking–Keel on Looijenga pairs:



Blow up p, q, r times of these three pts.

$\rightsquigarrow$ Y = resulting blow-up.



GHK: $\mathcal{T}_{p,q,r} \longleftrightarrow Y \setminus D$ proper transform of toric divisor removed.

(in progress):

$$D^b \mathcal{W}(\mathcal{T}_{p,q,r}) \cong D^b Coh(Y \setminus D).$$

v. cycles $\longleftrightarrow$ spherical objects.

$i_* \mathcal{O}_E$ spherical sheaves, $i_* \mathcal{O}_C$,

$i_* \mathcal{O}_C(-1)$. two spherical objects even though one line.

Thm: $\exists$ exact Lag'n torus T.

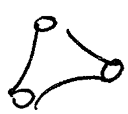
- Pick spin str.
- Flat cplx. line bdles on $T \Longleftrightarrow (\mathbb{C}^*)^2$ choices.

$\rightsquigarrow (T, (\alpha, \beta)),$

$$\begin{tikzcd} & (\mathbb{C}^*)^2 \subset \mathbb{P}^2 \\ \arrow[r, leftrightarrow] & (\mathbb{C}^*)^2 \subset Y \arrow[u] \\ (T,(\alpha,\beta)) \arrow[r, leftrightarrow] & \text{pt.} \in (\mathbb{C}^*)^2 \longleftrightarrow \mathcal{O}_{pt}. \end{tikzcd}$$

Properties: V van. cycle. some cases, true for any Lag'n sphere.

$$HF((T, (\alpha, \beta)), V) = 0$$

for $(\mathbb{C}^*)^2 \setminus$ 

$(\mathbb{C}^*)^2 \cap$ chain of -2 curves $= (\mathbb{C}^*)^2 \cap \tilde{\Sigma}$ ← original curve proper transform.

$=$  . .

Consequence: $HF^*((T,\alpha,\beta),(T,\alpha,\beta)) = H^*(T^2) \neq 0,$

so $\Rightarrow$ vanishing cycles don't generate $\mathrm{Fuk}(T_{p,q,r})$.

Contrast (Seidel): weighted homogeneous singularities, weights w_i w/ $\sum \frac{1}{w_i} \neq 1$.
~~where~~ Fukaya category here is generated by vanishing cycles.
(Have examples of these tori where $\sum(\frac{1}{w_i}) = 1$, so condition's strict!)

Many more examples of exact Lag'n tori: $\rightsquigarrow$ some still corresp. to $(\mathbb{C}^*)^2$ patches
(e.g. verify the slogan
"exact Lag'n T^2 corresponds to a chamber of a scattering diagram."
in simple examples.)

$\rightsquigarrow$ ⚠ $\exists$ plenty of tori that are NOT mirror to $(\mathbb{C}^*)^2$ patches
(expected from HMS).

Q: Do chambers correspond to cluster coord. charts?

A: In principle, yes.

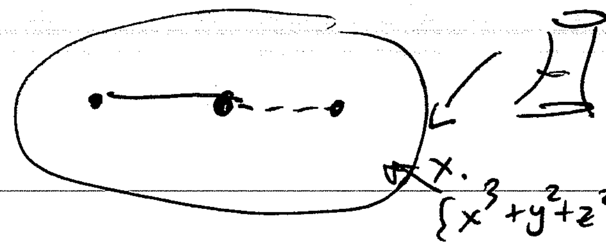
Automorphisms of $\mathcal{T}_{p,q,r}$ (exact, cptly supported symplectomorphisms).

Construct automorphisms of a Lefschetz fibration:

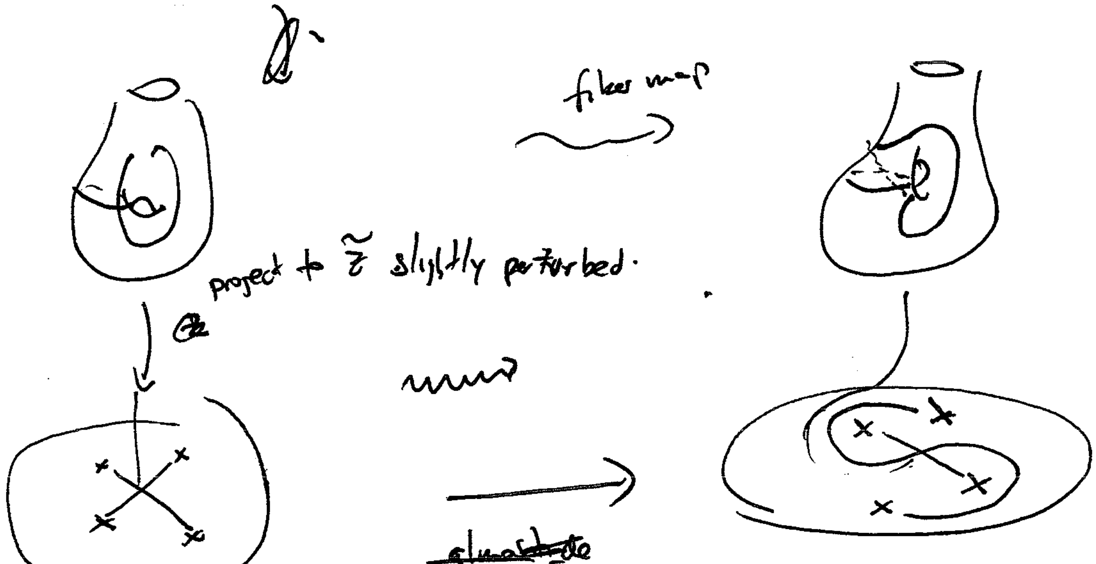
- automorphism of ~~base~~ fibre +
- automorphism of the base.

} need to be compatible. + cptly supported.

Ex. • Dehn twist in A_n



• A_2 again:



~~almost~~ Matching up base almost ~~mat~~ gives out of base.

(add non-cpct arc.

Check maps are compatible; new lines upstairs are vanishing paths.

Claim: this is a Dehn twist.

C.f.: [Giroux–Pardon] ⟹ any Milnor fibre is the total space of a Lefschetz fibration, but not necessarily that any aut. is fibered.

Might hope: thanks to [Auroux–Muñoz–Presas] that any automorphism is fibered, after stabilizing.

Thm: $\exists$ exact, cpctly supported symplectos of $\mathcal{J}_{p,q,r}$ which are not isotopic to a product of Dehn twists
[in vanishing cycles some ~~same~~ p,q,r.
[any spheres other p,q,r.

- constructed as aut of Lefschetz fibrations
using similar pictures, but orders of magnitude more complicated.

- Contrast: • Riemann surfaces, MCG gen. by Dehn twist,
& • A_n Milnor fibres (J. Evans, W. Wu), gen. by twists.

Pf: uses exact Lag'n tori & the fact that they're almost always (homologically) disjoint from spheres.
in Fuk.

Ex: in one example, a torus T which corresponds to a $(\mathbb{C}^*)^2$ patch $\xrightarrow{\text{symplecto}}$ torus T' which can't correspond to a $(\mathbb{C}^*)^2$
b/c
$HF^*((T', \alpha, \beta), V) \neq 0$ some fixed V, (for all α, β).

If symplecto product of twists, could never map something to something

Q: What's the mirrors to these things? ~~Need to~~ Not some birational transformation, b/c doesn't preserve $(\mathbb{C}^\times)^2$ patches.