Kontsevich I, Jan. 2015 Holomorphic Log'n Freehomology

X cpct, Kähler.

$\rightsquigarrow$ on $H^\cdot(X)$ get a Hodge str., interaction of 3 theories:

Betti: $H_B^i(X) = H^i(X, \mathbb{Z})$

deRham $H_{dR}(X) = \mathbb{H}(X, (\Omega_X^\cdot, d))$.

Dolbeault: $H_{Dol}(X) = \oplus H^\cdot(X, \Omega^\cdot)$.

Related: $H_B^\cdot \otimes \mathbb{C} = H_{dR}$. H_{dR} has a Hodge filtration, & $H_{Dol} = gr_F H_{dR}$

Generalization:

X cpct Kähler + α hol. (closed) 1-form, so $d\alpha = 0$.

$H_{dR}(X, \alpha) := \mathbb{H}^\cdot(X, (\Omega_X^\cdot, d + \alpha \wedge \cdot))$.

holomorphic d

$H_{Dol}(X, \alpha) := \mathbb{H}(X, (\Omega_X, \alpha \cdot))$.

H_B: later.

Thm (?) $\dim_{\mathbb{C}} H_{Dr}(X, \alpha) = \dim_{\mathbb{C}} H_{Dol}(X, \alpha)$

(C. Sabbah)

(in fact, canonically isomorphic)

Proof: On $\Gamma(X, \Omega^\cdot_{C^\infty})$

$\bar\partial$ resolution of $\partial \oplus \alpha$

differentials. $D = \partial + \bar\partial + \alpha$ $= \bar\partial + (\partial + \alpha)$

$= d_{C^\infty} + \alpha$ connection form.

Calculates H_{dR}

• $D_2 = (\bar\partial - \frac{1}{2}\bar\alpha) + (\frac{1}{2}\alpha.)$ — Higgs field.

Straightforward calculation: $\nabla_{\bar\partial}$ for some holom. str. on ~~...~~ triv. $\mathbb{C}^\times \times X \to X$ ⇝ $\mathcal{L}$.

$$D_1 D_1^+ + D_1^+ D_1 = 2(D_2 D_2^+ + D_2^+ D_2)$$

Laplacians same, so D_1, D_2 have same cohomology (b/c same Harmonic forms!)

So, $$\ker D_2 / \mathrm{Im}\, D_2 = \mathbb{H}^\cdot(X, (\underbrace{\mathcal{L}}_{\text{holom. line bdle.}} \otimes \Omega^\cdot, \tfrac{\alpha}{2}\wedge))$$

$$\neq \mathbb{H}^\cdot(X, (\Omega^\cdot, \tfrac{\alpha}{2}\wedge))$$

$$= \bigoplus H^i(\text{neigh of } Z_i, (\Omega, \alpha_i)).$$

$$\cong \bigoplus H^\cdot(\text{neighborhood of } Z_i, (\mathcal{L} \otimes \Omega^\cdot, \alpha))$$

$$\mathrm{Zeros}(\alpha) = \coprod_{\text{conn. components}} Z_i$$

$[\alpha_i]|_{Z_i} = 0 \in H^2(Z_i, \mathbb{C})$.

$\alpha = df_i$ in nhood of Z_i.

Canonical after normalization $f_i|_{Z_i} = 0$.

Generalization of degeneration of HdR spectral sequence.

At the moment $\nexists$ a Deligne–Illusie type char. p. proof of the above generalization!

<u>More generally</u>:

$$H^\cdot(\Omega^\cdot_{C^\infty}(X), d_{C^\infty} + z_1\alpha + z_2\bar\alpha) \simeq H^\cdot_{Dol}(X, \alpha) \text{ for}$$
$$(z_1, z_2) \in U \subset \mathbb{C}^2, \text{ w/ } z_2 \neq -\bar z_1.$$

(same argument!)

Question:

Consider trivial bdle.: $U \times H_{Dol}(X, \alpha)$ triv.

this is a hol. bdle, w/ fiber H_{Dol} (that depends on z_1, z_2).

$$\begin{array}{c} U \times H_{Dol}(X,\alpha) \\ \downarrow \\ U. \end{array}$$

holom. str.

$\nabla_{\frac{\partial}{\partial z_1}}, \nabla_{\frac{\partial}{\partial \bar{z}_2}}$ commuting operators on space of sections

$$\frac{\partial}{\partial z_1} + \boxed{}$$

matrix valued fcns.

(these values over (z_1, z_2) generate some Lie algebra ... proposed by many?)

In general: the thm.

(**Thm ①** $\dim H_{dR}(X, \alpha) = \dim H_{Dol}(X, \alpha)$)

The same is true, sometimes, when X is non-cpct, algebraic / $\mathbb{C}$

use Zariski topology

Sufficient condition: $X = \overline{X} \setminus D = \bigcup D_i$, s.n.c.-divisor.

locally $\alpha = \sum c_i \frac{dz_j}{z_j}$ (in analytic topology) $+ df$

(+ differential of some functions)

Assume $f = \underset{\neq 0}{\text{const.}} \prod_j \frac{1}{z_j^{d_j}} (1 + O(1))$. ~~all $d_j \geq$~~, all $d_j \geq 1$..

$\rightsquigarrow$ no crit. pts. near divisors $\rightsquigarrow$ zeroes(α) cpct

(Need to use Hodge theory 6 connections w/ irregular. Singularities

(Special case: $\alpha = dw$, w proper. à la (K-Pantev) Mochizuki's work for this case).

Thm. ① $\longleftrightarrow$

$$H_\hbar = \mathbb{H}^\cdot(X_{Zar}, (\Omega^\cdot, \hbar d + \alpha))$$

$\hbar \in \mathbb{C}$ planck const.

Then, $(H_\hbar)_{\hbar \in \mathbb{C}}$ alg. vector bdle $/ \mathbb{C}_\hbar$ (NOT coherent sheaf, ~~[illegible]~~)

fiber at $\hbar \neq 0$ $H_{dR}(X, \frac{\alpha}{\hbar})$

$\hbar = 0$ $H_{Dol}(X, \alpha)$.

? $H_B^\cdot(X, \alpha)$?

$\mathbb{C}$-vector space: $H_B^\cdot(X_{an}, \mathbb{C}^*$ local system given by $d_{\mathbb{C}^\infty} + \alpha)$.

Claim: for generic $\hbar \rightsquigarrow \mathbb{Z}$-lattice.

(Generic means $\hbar \notin$ Stokes rays (countably many of them)).

How to proceed? Take:

$$\widetilde{X}^{ab} \xrightarrow[pr]{} X$$

universal abelian cover or smaller w/ fiber $H^1(X, \mathbb{Z}) / \ker(\int \alpha : H_1(X, \mathbb{Z}) \to \mathbb{C})$

On $\widetilde{X}^{ab}$, $pr^* \alpha = dF$ holom. fcn.

$\rightsquigarrow$ countably many critical values $\in \mathbb{C}$.

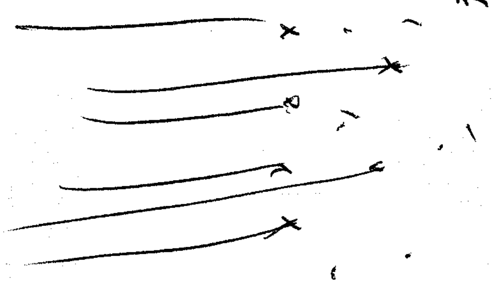
Stokes rays are straight lines connecting critical values here;

$= \mathbb{R}_{\geq 0}(w_1 - w_2)$ $w_1 \neq w_2$ are crit. values of F

(defined up to const., so $w_1 - w_2$ well defined).

$\mathbb{R}_{\le 0}$ + crit. values of $\frac{\alpha}{\hbar}$, not Stokes rays

do not intersect each other, so (up to rotation):



Recall $\mathrm{Zeroes}(\alpha) = \coprod Z_i$ connected components

compact.

Near Z_i, $\alpha = df_i$ $f_i|_{Z_i} = 0$.

$\rightsquigarrow$ "sheaf of vanishing cycles"

$$H_{B,i,\hbar} = R\Gamma\left(Z_i, \psi_{\frac{f_i}{\hbar}}(\mathbb{Z}_X)\right)$$

$$:= H^{\bullet}\left(U_\varepsilon(Z_i), \left(\tfrac{f}{\hbar}\right)^{-1}(\eta); \mathbb{Z}\right)$$

(Milnor type def'n; but ∃ more abstract def'ns).

, where $0 \le \eta \le \varepsilon^{\forall}$ (e.g. $\eta = e^{-\frac{1}{\varepsilon}}$).

(claim this stabilizes for very small ε, so get $\mathbb{Z}$-lattice.)

Get canonical iso

$$H^{\bullet}_B(X, \mathbb{C}^* \text{ local sys.}) \simeq \mathbb{C} \otimes \Big(\bigoplus_i H_{B,i,\hbar}\Big)$$

integral lattice.

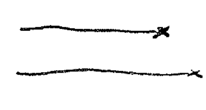
Assume all crit. points isolated & holom. Morse type (general case slight generalization)

$\rightarrow$ gradient lines of $\mathrm{Re}\,F$ on X^{ab}

(Kähler metric) $\xrightarrow{F}$ Horizontal lines in $\mathbb{C}$.

$\{\mathrm{Im}\,F, \cdot\}$

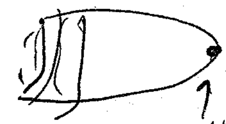
Ham. flow for $\mathrm{Im}\,F$.



"thimbles."

get thimbles $\approx \mathbb{R}^n$, $n = \dim_{\mathbb{C}} X$.

Along each thimble, trivialize local system,



use apriori trivialization here, & parallel transport

$\rightsquigarrow$ get non-cpct. chain w/ coefficient in local system, pairing w/ any class in DeRham cohomology H_{dR}

class is $\leftrightarrow$ C^∞ form ω w/ $(d+\alpha)\wedge\omega = 0$

$\Longleftrightarrow$ $\left(d(e^F pr^*\omega) = 0\right)$

$\int_{\text{thimble}} e^{F/\hbar} pr^*(\omega)$ using $F|_{\text{pt}} = 0$ normalization

is absolutely convergent (not obvious, maybe only true for small $\hbar$; a priori depends on m'fld & form.)

(B/c volumes of vanishing cycles grows by some dynamics/vec. field, has some bound; relates to entropic stuff.)

Requires analytic work here to make it rigorously.

<u>Replacement of Hodge structure:</u>

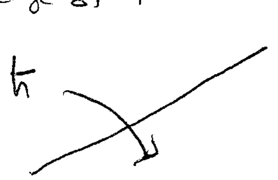
- Alg./hol. bundle / $\mathbb{C}_\hbar$.

for $\hbar \notin$ countable union of Stokes rays, get $\mathbb{Z}$-lattice.

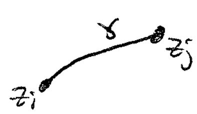
When we cross Stokes lines?

Case of Morse singularities: $\pm \tau_{i,\hbar}$ basis of thimbles in $(H_B)^*$ multiply by normalization (exponentially small multiple of interval along stokes ray)

Stokes line:

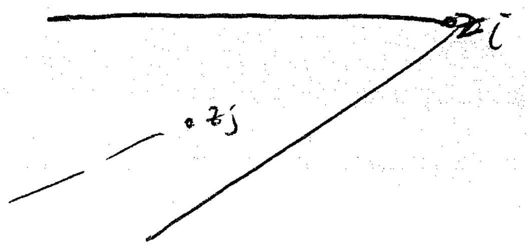


Then $\tau_{i,\hbar} \longrightarrow \tau_{i,\hbar} + \sum_{j \text{ crit. pts.}} \tau_{j,\hbar}\, e^{-\frac{1}{\hbar}\int_\gamma \alpha}\, n_{i,j,\gamma} \in \mathbb{Z}$, integers.



$\gamma \in H_1(X, \{z_i \cup z_j\}, \mathbb{Z})$ $\partial\gamma =$ pt. in z_i − pt. in z_j.

(j could = i).



Example: $X = \mathbb{C}^*_z \quad \alpha = \frac{dz}{z} - dz.$

At $z=0$, doesn't satisfy the assumptions, but still seems ok.

$\mathrm{Zeros}(\alpha) = \{1\}.$

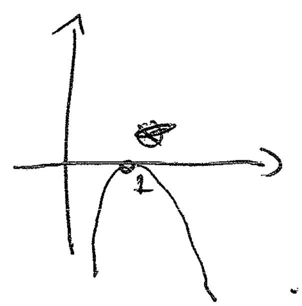
If $\hbar > 0$: know thimble: $\{z \in \mathbb{R}_{>0}\} \subseteq \mathbb{C}$.

$$\alpha\Big|_{\mathbb{R}>0} := d(\underbrace{\log z - z + 1}_{F}).$$

for $\beta = \frac{dz}{z}$, note

$$\left(d_t - \frac{\alpha}{\hbar}\right)\beta = 0.$$

& graph of F



$\rightsquigarrow$ pairing β & thimble get

$$I_+(\hbar) := \int_0^\infty e^{F/\hbar} \frac{dz}{z} = \int_0^\infty z^{1/\hbar} e^{-z/\hbar} e^{1/\hbar} \frac{dz}{z}.)$$

Take $z = \hbar t$: get

$$= \hbar^{1/\hbar} e^{1/\hbar} \underbrace{\int_0^\infty t^{1/\hbar} e^{-t} \frac{dt}{t}}_{\Gamma(\frac{1}{\hbar})}.$$

so $I_+(\hbar) = \hbar^{1/\hbar} e^{1/\hbar} \Gamma(\frac{1}{\hbar})$ for $\mathrm{Re}(\hbar) > 0$.

Can calculate:

$$\cancel{\text{Do}}\ I_-(\hbar) := \frac{2\pi i \hbar}{I_+(-\hbar)}$$

$\mathrm{Re}\,\hbar < 0$.

$$\tau \longmapsto \tau - \tau e^{-\frac{\int_\gamma \alpha}{\hbar}}$$

↑ b/c no other thimbles..

($n_{ij,\gamma} = 0, \pm 1$)

On imaginary line, limiting values; compare:

$$I_+(\hbar) = I_+(\hbar)\begin{cases} (1 - e^{-2\pi i/\hbar}) & \hbar \in \mathbb{R}_{>0} \\ (1 - e^{+2\pi i/\hbar}) & \hbar \in i\mathbb{R}_{>0} \end{cases}$$

Still something to understand, b/c this goes beyond usual assumptions..

In general:

(actually, a local system over $\mathbb{C}^\times_\hbar$, so non-trivial monodromy.)

graded Lie algebra / $\mathbb{Q}$. $\mathfrak{g} = \bigoplus_\gamma \mathfrak{g}_\gamma \quad \gamma \in H_1(X, \text{zeros}(\alpha), \mathbb{Z})$

corresponding (pro nilp.) group / $\mathbb{Z}$.

$\mathfrak{g}_\gamma \neq 0$ only if $\partial\gamma = \text{pt. in } Z_i - \text{pt. in } Z_j$.

$= \mathrm{Hom}(H_{B,i,\hbar}, H_{B,j,\hbar}) \otimes \mathbb{Q}$.

The jump data at Stokes rays

→ Wall crossing structure / wall crossing formula.

(satisfies support property --

& $n_{ij\gamma}$ grow no more than exponentially!)

Suppose $(X_b, \alpha_b)_{b \in B = \text{base}}$. & $\alpha_b = \alpha_{\text{global closed form}}\big|_{X_b}$.

$\rightsquigarrow$ get wall crossing structure on

$$B \times \mathbb{C}^*$$

(BPS in all directions

$\rightsquigarrow$ family of local systems on B depending on $\hbar$,
together w/ holomorphic limit ~~~~ at $\hbar = 0$
(Higgs bdle automatically at limit)).