Kontsevich II

Last time:

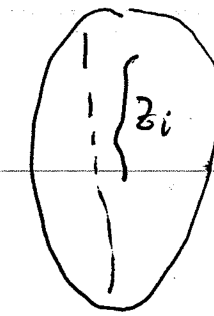
$(X, \alpha) \rightsquigarrow \left(H_{dR}(X, \frac{\alpha}{\hbar}),\ H_B(X, \frac{\alpha}{\hbar}),\ H_{Dol}(X, \alpha) \right)$ $\quad \hbar \in \mathbb{C}^\times$.

(α: hol. 1-form, $d\alpha = 0$)

$$\mathrm{Zeroes}(\alpha) = \bigcup Z_i \quad (Z_i \text{ conn. components})$$

near Z_i, $\alpha = df_i$.

$$\Rightarrow \quad H^\bullet_{B,i,\hbar} := \mathbb{H}^\bullet\left(Z_i, \psi_{\frac{f_i}{\hbar}}(\mathbb{Z}_X)\right)$$



$\hbar \in \mathbb{C}^\times$

local systems over $\mathbb{C}^\times_\hbar$.

$H_{dR}(X, \frac{\alpha}{\hbar})$ for $\hbar \neq 0$. "local data, no global computation" $H^\bullet(\text{neighborhood}, \frac{f}{\hbar}^{-1}(-\delta), \mathbb{Z})$.

$\hbar \notin$ Stokes rays $\xrightarrow[\text{can.}]{\sim} \left(H_{B,i,\hbar} \otimes \mathbb{C} \right)$

cross Stokes ray: jump $= id + \sum_{i,j} e^{-\frac{1}{\hbar}\int_\gamma \alpha}$ ("integr. operators? generates? operators? operators" $H_{Z_j, Z_i} \to H_{Z_i, i}$) unclear – look up last lecture

(~~normal~~ hmtl. class $\gamma \neq 0$ of path. count)

Extend to $\hbar = 0$ using

$$\bigoplus_i \mathbb{H}^\bullet\left(\text{neighb of } Z_i, \Omega[[\hbar]], \hbar d + \alpha\right)$$

"Hodge filtration"

<u>Observation:</u> Many things make sense in ∞-dimensions.

X ∞-dim'l complex.

No sense!

$$H^{\frac{\infty}{2}+\cdot}(X, (\Omega, d+\alpha))$$

$$H_B^{\frac{\infty}{2}+\cdot}\left(X, \begin{matrix}\text{local system}\\ d+\alpha\end{matrix}\right)$$

__Good case__: when zeroes = finite $\cup$ of finite dim'l analytic spaces.

(often happens; e.g. fredholm situation).

$\rightsquigarrow$ $\partial^2 f_i$ is __fredholm__

(can throw away ∞'ly many coords like "stabilization by quadratic form." & finite-reduce.)

Remarks: some sign ambiguity depending on even/odd ∞ # of reduction.
"__orientation data in DT theory__" needed to fix this.

In good cases, we have a construction of $H_\cdot(X, \alpha)$ (B, dR, Dol).

As in finite dim'l situation,

For (X_b, α_b) hol. family $\rightsquigarrow$ hol. family of cohomology over $B \times \mathbb{C}_\hbar$

Moreover if family __isomonodromic__ $\longrightarrow$ flat connection along B.

(meaning on total space have global closed hol. 1-form restricting to fibers α^{global}, $d\alpha^{global} = 0$, $\alpha_b = (\alpha^{global})|_{X_b}$.)

__Main example__: (M, ω) $\omega^{2,0}$ fin. dim'l complex symplectic.

$L_l, L_r \subset M$ holom. Lag's. $X := \text{Maps}\left([0,1] \xrightarrow{\varphi} M \;\middle|\; \begin{matrix}\varphi(0) \in L_l\\ \varphi(1) \in L_r\end{matrix}\right)$

$\alpha = \int_{path} \omega$. $\mathbb{C}$-valued hol. 1-form. "gradient for real part..."

Zeroes of this form are

$$\mathrm{Zeros}(\alpha) = L_\ell \cap L_r .$$

$\rightsquigarrow$ " $HF^\bullet_\hbar(L_\ell, L_r)$ ", plus integral str. for lot of points.

gradient lines = pseudoholomorphic discs for "rotated" almost complex structure.

Version: $H^\bullet(X, \Omega_{C^\infty}, d + \frac{\alpha}{\hbar} + \overset{\text{correction}}{R\hbar\bar{\alpha}})$

to get family of cohomology over $\mathbb{CP}^1_\hbar$. (for $R >> 0$).

"twistor version".

Main example: $M = T^*Y$,

$L_r \subseteq M$. $\quad L_\ell = T^*_y \quad y \in Y$.

$\rightsquigarrow HF^*_\hbar(T^*_y, L) \quad \hbar \neq 0$ family depending on y.

isomonodromic family (deform y; exact deformations).

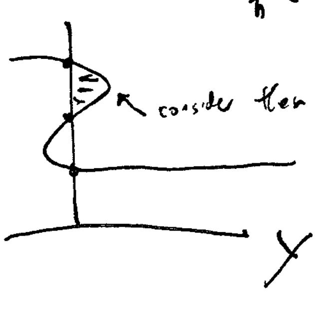
$\Rightarrow$ get flat connection, depending on $\hbar$.

Claim: exactly Gaiotto-Moore-Neitzke sol'n of Hitchin system (family of flat connection --),

Use Def'n: $\hbar \notin$ Stokes ray, then

$$HF_\hbar(T^*_y, L) = \Big(\bigoplus_{T^*_y \cap L} \mathbb{Z}\Big) \otimes \mathbb{C} .$$

(intersection pts)



consider them along Stokes rays $\leftrightarrow$ J hol. disks. for rotated J.

Take Limit: (stable under deform) make cotangent bundle very small

$\rightsquigarrow$ projects to trees (on the nose).

Follows from general principle. "fiber of local system is middle dim'l cohomology of space of paths."

Conjecture (M.K., A. Belov-Kanel):

Aut (Diff'l operator on $\mathbb{A}^n_{\mathbb{C}}$) $\leftarrow$ $\mathbb{C}\langle p_i, q_i\rangle / ([\hat{p}_i, \hat{q}_i] = 1$, rest $[\,,\,] = 0)$ — Weyl alg.

$= \mathrm{Aut}\,(\mathbb{C}^{2n} = T^*\mathbb{A}^n$, with $\sum dp_i \wedge dq_i)$.

Argument used reduction to prime #'s / mystery. "says # D-modules = # hol. Lag'ns"

More general conj:

$L \subset \mathbb{C}^{2n}$, $H_1(L, \mathbb{Z}) = 0$

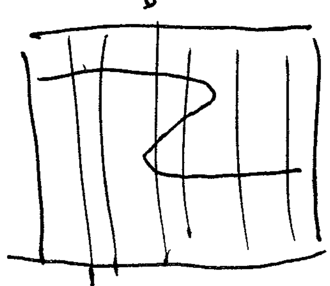
alg. Lag'n. - - - -> canonical holonomic $D_{\mathbb{A}^n}$ module associated to it (not homogeneous).

(why more general? Given $\varphi : \mathbb{C}^{2n} \circlearrowleft$ sympl., take Γ_φ lag'n).

** This construction gives a candidate: what is the D-module associated to L!

(Maybe smoothness not relevant, but just) exactness ?? smoothness of L used, but L ?).

what is it?



$\mathbb{C}^{2n} \simeq T^*\underbrace{\mathbb{C}^n}_{Y}$

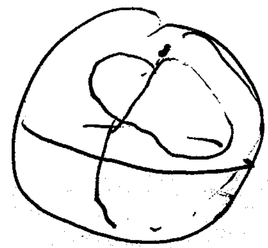
$HF_{\hbar=1}(T^*_y, L)$ is a local system.

on $\mathbb{C}^n$ (w/ no singularities up to sympl. rotation at zero section $\Rightarrow$ Vec. bdle.)

But irregular: $\exists$ Stokes str.

"Stokes str. at ∞"

At ∞: S^{2n-1} ⟶



S^{2n-1}

L = graph of multivalued fcn. dF.

Q: when are real parts of these ~~branch~~ fcns. the same (different?) — open locus; walls

partitions S^{2n-1} higher-D. generalization of Stokes filtration (language maybe introduced by Kashiwara & Schapira).

when
$[p\,dq|_L] = 0 \in H^1(L)$

⇝ α on path space is exact

⇝ finite calculation (finitely many rays, etc.)

" Q: how to go back from D-modules to Lag'ns?" (Not sure, but ought to be easier.)

" this process Lag'ns → D-modules is faithful. ~~It's hard~~ Open Q. was to construct Lag's from D-modules.."

get (Lag'ns w/ bdles should give all D-modules).

Not only holonomic D-module, but exponential Motivic hol. D-module; e.g. all D-modules of the form

$$Z \xrightarrow{p_2} \mathbb{A}^1$$
$$\downarrow p_1$$
$$Y$$

+ (closed) vale correspondences.

$(p_1)_*(p_2^* \exp)$. — exponential D-module..

Uses transcendental Riemann Hilbert corresp.

$\overline{Y}$ cpt. ⇝ T^*Y, & $T^*Y \cup \mathbb{P}T^*Y$ — compactification.. contact boundary

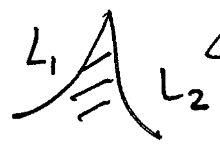
<u>Support</u>: more general def'n:

~~$\Lambda \subset T^*Y$~~ irred. Leg. subvariety.

$$= \{ (\Lambda \subset \mathbb{P}T^*Y, \text{ germ of } L, \text{ closure} = \Lambda \cap \mathbb{P}T^*L) \}.$$

two germs L_1, L_2 equivalent if try to integrate two form over, its logarithmically divergent integral.

(truncate Puiseux series, keep fin. many coeffs. . .



support is a germ <u>keep more than conical behavior at the point</u>.

<u>Multiplicative version</u>:

$(\mathbb{C}^*)^{2n}$ w/ form $\sum d\log x_i \wedge d\log y_i$.

comes from class $\in K_2\left(\begin{matrix}\text{rational functions} \\ \sum [p_i] \wedge [q_i]\end{matrix}\right)$,

$L \in (\mathbb{C}^*)^{2n}$ (Lag'n manifold), if when restrict class to L in K_2? get 0.

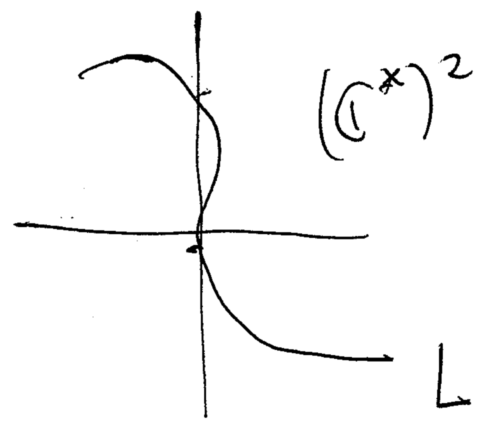
from cluster coordinates, get K_2-Lagrangians

$\rightsquigarrow$ holonomic module over quantum torus (canon.)

$$\hat{X}_i \hat{Y}_i = q Y_i X_i . \quad q \in \mathbb{C}^*$$

$$q = e^{1/\hbar}.$$

E.g.



$\rightsquigarrow HF^*(\{x\}\times(\mathbb{C}^*)^n, L)$ ← Y var.

$\forall x \in (\mathbb{C}^*)^n$.

Not isomonodromic b/c coh. class changes, (assuming convergence). ← when you deform, get 1-form on $(\mathbb{C}^*)^n$, cohomology NOT $\mathbb{Q}$.

$\rightsquigarrow$ hol. family. get bdle over $\mathbb{C}^*$.

Conj: Consider algebra gen. by

$$\hat{X}^{\pm 1}, \hat{Y}^{\pm 1} / \hat{X}\hat{Y} = q\hat{Y}\hat{X}. \quad q \in \mathbb{C}^*, \; 0<|q|<1.$$

& holonomic module which is an algebraic vector bundle / $\mathbb{C}^*_{\hat{X}}$

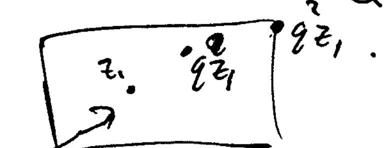
equivariant w.r.t. $\hat{X} \to q\hat{X}$. dilation by powers of q.

Conj: This is the same as

$\Longleftrightarrow$ holom. vec. bdle on $E = \mathbb{C}^*/q$, plus 2 splittings of slope filtration (filtration by rat'l #'s $\mathbb{Q}$). (Harder-Narasimhan).

analogue of RH corresp. for quantum D-modules.

Idea: compose linear maps; eigenvalues like Lyapunov exponents $\rightsquigarrow$ get filtration



$L_x = \{x\} \times (\mathbb{C}^\times)^2$, why $L_x = L_{qx}$. ??
(don't intersect).

Consider two-param. family

$$L_{x,w} = L_x \text{ w/ local } \mathbb{C}^* \text{ system w/ monodromy mult. by } w \in \mathbb{C}^*.$$

$\rightsquigarrow HF^\cdot(L_{x,w}, L)$

So on ~~$(\mathbb{C}^*)^2$~~ $\mathbb{C}^* \times \mathbb{C}^\times$ foliation on which cohomology doesn't change.

Claim: $x\frac{\partial}{\partial x} = \frac{1}{\hbar} w\frac{\partial}{\partial w}$

↑ shifting by x ⟷ ↑ by shift by w.

$x \longrightarrow e^{2\pi i \hbar} x$ are the same leaf.

Meaning one can make these equivalent...?

Guess: For K_2-Lagr'n manifolds,

write $q = \exp(2\pi i \tau)$ $\operatorname{Im} \tau > 0$ $(\tau \in \mathbb{H})$. (upper half)

R-H
$\rightsquigarrow$ holom. vec. bdles $/ \mathbb{C} / \mathbb{Z} + i\tau\mathbb{Z}$

+ 2 filtrations holomorphically depending on things

Maybe equivariant w.r.t. $SL(2, \mathbb{Z})$ (b/c modular trns. gives same elliptic curve).

alg. family of q-D-modules.

Q: is it ~~the~~ same for ~~same~~ vec. bdle? (if so, explains why modular forms pop out)
(e.g. are they identified by modular examples?)