Kontsevich III:

X kähler $\omega_X^{1,1}$

- hol. vol. form Ω_X (not necessarily (strong) $C{-}Y$; doesn't satisfy Kähler–Einstein eq'n)

& $X \xrightarrow{\pi} Y$ kähler. $\dim Y < \dim X$, some singular fibers, etc.

$\rightsquigarrow \mathcal{F}(X, \frac{1}{\hbar}\omega_X^{1,1} + T\pi^*\omega_Y^{1,1})$ $\mathbb{Z}$-graded category.

? with a stability structure ? (we expect).

Over which field? k any field $\rightsquigarrow$ $N(k)$ = Novikov field

$= \left\{ \sum_{\infty} a_{\lambda_i} t^{\lambda_i} \;\middle|\; a_{\lambda_i} \in k,\ \lambda_i \in \mathbb{R},\ \lim \lambda_i = +\infty \right\}$.

over $\overbrace{N(\underbrace{N(\mathbb{Q})}_{\omega_X \text{ factor}})}^{\downarrow\, Y \text{ factor}}$! (two param family)

(idea: T very large $T \gg$ any ~~power~~ fxn. of $\hbar$.) $\hbar_X \cdot T \to \infty$

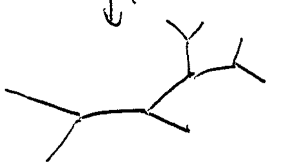
$\hbar_X \to 0$.

Expected Semistable objects:

- special Lag'ns in X w.r.t. form + local system, —

s.t. fiber special Lag'n in fiber. $\longrightarrow$ $\mathcal{L}_X \subset X$

$\downarrow$ projects to



singular Lag'n $\mathcal{L}_Y \subset Y$.

On $Y \setminus \mathrm{Disc}$: local system of CY categories $\mathcal{F}_y = \mathcal{F}(\pi^{-1}(y), \omega_X^{1,1} + T\pi^*\omega_Y)$

should carry stability w/ values in

$$\mathbb{C}\text{-line} \quad K_y \simeq \mathbb{C}$$
$$\|$$
$$\det(T_y^* Y).$$

(In general, one should have a "perverse sheaf of categories at disc."
as in Kapranov's work".

& s.s. objects in $\mathcal{F}(X..)$ slope θ

= sing Lag'ns $L_Y \subset Y$ + objects $\mathcal{E}_y \in \mathcal{F}_y$ for some $y \in \mathcal{L}_Y$.

w/ $\mathrm{Arg}(Z(\mathcal{E}_y)|_{\mathcal{L}_Y}) = \theta$.

Idea: One can try to do this for such local systems of categories non-geometrically

Special case $\dim Y = 1$. curve.

Precise prediction: Suppose Y = elliptic curve, e.g. $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, &
has $\mathrm{vol}_Y = dz$; plus
"arbitrary real symplectic form ω_Y" (not nec. constant).

$\mathcal{C}$ $\forall$ A_∞ category/ $\forall$ field k + stab. str., e.g. "case of trivial local system"

$$k_0(\mathcal{C}) \to \underset{\text{fin. rank}}{\Gamma} \xrightarrow{Z} \mathbb{C}.$$

$\leadsto$ new category $\quad \tilde{\mathcal{C}}/N(k) \quad (\mathcal{C} \otimes_k N(k)) \underset{N(k)}{\otimes} \mathrm{Perf}\Big(\overset{\text{roughly}}{\text{Tate elliptic}}\ N(k)^\times/t^{\mathbb{Z}}$

$t = \exp(-\text{Area torus})$.

& $k_0(\tilde{C}) \longrightarrow \tilde{\Gamma} = \cancel{\Gamma \otimes \mathbb{Z}^2 \xrightarrow{Z_c \otimes \int} \mathbb{C}}$

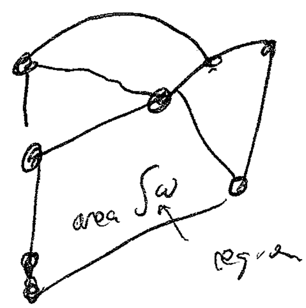
$$\tilde{Z}_C : \quad \tilde{\Gamma} = \Gamma \otimes H_1(Y, \mathbb{Z})$$

$$\begin{array}{ccc} \big\downarrow Z_c & & \big\downarrow \int v\, d\partial y \\ \mathbb{C} & \otimes & \mathbb{C} \end{array}$$

$$\big\downarrow$$

$$\mathbb{C}.$$

Description of $\tilde{C}$:



$$G \subset Y = (S^1)^2$$

finite (ribbon) graph, piecewise linear/real analytic.

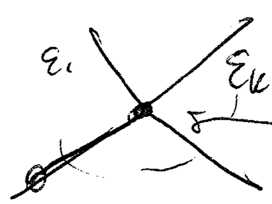
Assume $Y \setminus G$ = union of discs, each w/ area.

on G, put objects of C w/ some gluing condition:

$(\varepsilon_1, \ldots, \varepsilon_k)$, come from

$\mathrm{Rep}(A_{k+1}) \to C.$

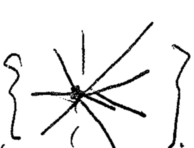
e.g.



the zeros from $\mathrm{Rep}(A_{k+1})$, e.g. triangle

e.g. $\bullet \to \bullet \to \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$.

$\Rightarrow$ two flags



$$0 \to E'_{\leq 1} \hookrightarrow F_{\leq 2} \hookleftarrow \cdots F''_{\leq 1} \leftarrow F''_{\leq 2} \leftarrow F''_{\leq 3}$$

ways to put object of C on B

$\longleftrightarrow$ functors: cats of finite type

$$F(G) \longrightarrow C$$

"Fukaya category over tubular nbhood of the graph."

~~The areas of discs gives a deformation of this category over Novikov field.~~

Supposed to be absolutely canonical objects. (solve M-C eq'n on graph! hol. discs = embedded polygons.)

$\tilde{F}_q(G)$ depends on graph $\rightsquigarrow$ ~~A~~ real A_∞ category.

larger graph: get different category, can take inductive limit.

$\rightsquigarrow$ $\tilde{C} = \varinjlim_{\substack{\text{all real} \\ \text{analytic graphs}}}$ of deformed categories $\mathrm{Fun}(F(G), C)$.

Ex: plane $\mathbb{R}^2$, $G = \mathbb{R}$, real line.

$C/_k$ ($\mathbb{Z}$-graded) : object of deformed category is an object

$\mathcal{E} \in C$, + $\gamma \in \mathrm{Hom}^1(\mathcal{E},\mathcal{E}) \hat{\otimes} N_+(K)$.

Satisfying MC eq'n. gives category over ring of integers of Novikov field, $N_{\geq 0}$ then localize.

(Not clear get triangulated category; when localize get extra objects in general!

"localize"

$\rightsquigarrow \mathrm{Hom}(\ \ ) \otimes N(k)$.

~~Precise~~

Prediction:

$\mathcal{C}_\theta = \{$ objects in $\mathcal{C}$ | Graph is piecewise linear, & classical limit gives ~~a~~ a system of objects of C on edges of G.

These objects $\mathcal{E}_e$ are semistable, & slope of edge $=$

$\mathrm{slope}(e) = \theta - \mathrm{slope}(Z(\mathcal{E}_e))\}$.

Conj: There is a stability structure & the category is triangulated.

No statement about MC elt; completely free in stability (to do what they want)

Easy

Thm: If C A_∞ cat. w/ stability str.

$\rightsquigarrow$ ~~heart~~ $\mathcal{A} = \mathrm{heart}(C)$ abelian.

Then, deformed objects in the heart form again an abelian category over $N(k)$ (~~???~~ after localizing scalars).

Really need stability str. for this..

Can iterate this construction $\rightsquigarrow$ get perfect cplxes. over ~~of~~ abelian variety over iterated Novikov field, & complete list of objects.

gives a description of stability

"networks/tropical curves; all stable slope θ, sum slopes vanishes, so balancing condition.:"

Conclusion

$\rightsquigarrow$ get stability str. on

$$\mathbb{C}^*/q_1^{\mathbb{Z}} \times \mathbb{C}^*/q_2^{\mathbb{Z}} \times \cdots \times \mathbb{C}^*/q_n^{\mathbb{Z}} \Big/ \mathbb{Q}((q_1))((q_2))\cdots((q_n))$$

$\Gamma = (\mathbb{Z}^2)^{\otimes n}$, $Z : \Gamma \to$

$$Z = \bigotimes_{i}^{n} Z_i \qquad Z_i : \mathbb{Z}^2 \xrightarrow{\int \Omega_x} \mathbb{C}$$

analytic preserving.

my non-deg. for categories close to cusp of moduli space.

$\rightsquigarrow$ (fibered) PL Lag'n variety.

In general, one should not expect

(C, Z_C) cat w/ stability condition

$(D(\text{ell curve } E/\mathbb{C}, Z_E))$ "fixed parameter" elliptic curve

$\}$

$\otimes$ of such stability

K group could be larger than tensor product of K group!

(categorical stability is idempotent closed)

E.g. $\tilde{\Gamma} = \Gamma$

T torus $(S^1)^n$ flat sympl. str

+ $\mathbb{C}$-induced constant vol. ~~form~~ ~~hol~~ = volume holom.

$\rightsquigarrow$ get stability.

But,

$$\mathrm{vol} = \mathrm{Re}\,\mathrm{vol} + i\,\mathrm{Im}\,\mathrm{vol},$$

$$\rightsquigarrow \mathrm{vol}' = 2\,\mathrm{Re}\,\mathrm{vol} + i\,\mathrm{Im}\,\mathrm{vol}.$$

expect to still get stability b/c $(SL(2,\mathbb{Z}))$

~~Case~~ Claim: can multiply by T^n.

flat Kähler CY tori $\rightsquigarrow$ flat subtori on which total volume form vanishes. problem.

(No problem b/c in this talk, "generic ~~curve~~" & H_1 of motive & cannot interact w/ H_1 of category :-)

$\{\gamma \in \Gamma \mid \exists$ ss objects$\}$

$\rightsquigarrow$ natural 1-param. family $/ \otimes \times \mathbb{R}/4\mathbb{Z}$ of convex(?) cones in $\Gamma \otimes \mathbb{R}$. $\rightsquigarrow$ $v_1 \otimes \dots \otimes v_n$ one to the power $\rightsquigarrow \Gamma / \mathbb{Z}(v_i)$ $v_i \in \mathbb{R}^2$

Consider subset Γ

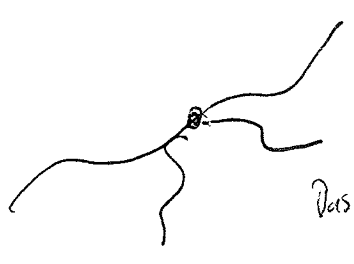
$v_1 \otimes \dots \otimes v_n$ of split vectors

$v_i \in \mathbb{Z}_i^2$. replace by $\mathbb{R}$

Get complete control of stability str.

Try to prove via mean curvature flow for graphs.

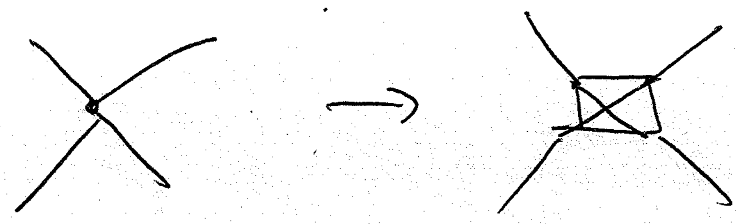
Also basic difficulty first:



mean curvature flow & trade bonds change

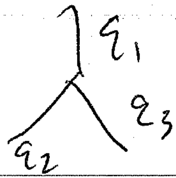
Basic difficulty: graph shouldn't stay same.

Eg.



Example:
(Fabian Haiden).
fiber cat
$= \mathrm{Rep}(D_4)$ derived.

b



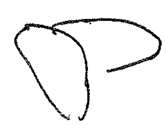
Novikov field.

Moduli space of such exact triangles in ~~Novikov~~ original field is $\mathbb{P}^1 - \{0,1,\infty\}$
for original field.

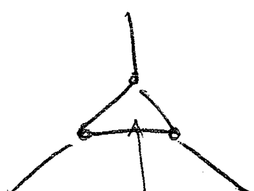
$\rightsquigarrow \mathbb{P}^1 \setminus \{0,1,\infty\}$ over $N(k)$ for $N(k)$.

Non arch: decompose:

$\coprod$ 3 open discs $\cancel{0<}\ |x| < 1$ + good reduction / 4th piece.



this case, replace by



new area of disc.

good case

General pattern which should just work for amoebas.

Some precise conjectures.

$\mathcal{C}$ cat. w/ stability.

2) distinguished polygons $\subset \mathbb{R}^2$ /shift

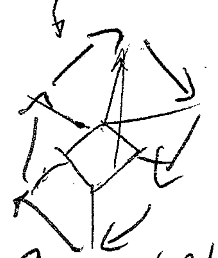
$\mathcal{E} \in \mathrm{Heart}(\mathcal{C})$

+ two finite filtrations anti Harder-Narasimhan.

e.g. quotients SS but slopes go wrong way,

Looking at central charges of objects:

one filtration, other:



gives polygon. (only so many such exist.)

each polygon
(should all come from other objects)

Technical condition: no sum of edges vanishes

for Consider any fxn. on boundary: Then fxn should extend to PL fxn. w.r.t. to some admissible sub polygons (?!)