Perutz, HMS & ~~the~~ closed-open string maps

with N. Sheridan

Phase 2: also w/ S. Ganatra

Setup: A side

- (X^{2n}, ω) cpt. sympl.
- $c_1(TX)$ trivialized
- $D \subset X$ codim-2 sympl. submfld (or more generally s.n.c. divisor)
- $\theta \in \Omega^1(X \setminus D)$ w/ $d\theta = \omega|_{X \setminus D}$.

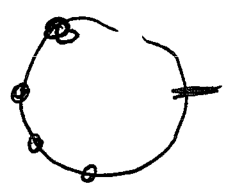
We'll be using the relative Fukaya category

$$\mathcal{F}(X, D) : \text{ curved } A_\infty \text{ cat. over } \mathbb{K} = \mathbb{C}((q)).$$

ob: cpt exact Lag'ns in $X \setminus D$.

hom: usual Floer cochain spaces, coeffs in $\mathbb{K}$.

str. maps $\{\mu^d\}_{d \geq 0}$ involve "pseudo-hol. polygons" weighted by $q^{u \cdot D}$.



$\mathcal{F} = tw^\pi \mathcal{F}(X, D)_{MC}$ split-closed triangulated A_∞ category.

B-side: $\check{\mathcal{X}}_{an}$ cx. analytic family, smooth, proper, rel. dim'n n.

$$\begin{array}{c} \check{\mathcal{X}}_{an} \\ \downarrow \\ \Delta^* \subseteq \mathbb{C} \end{array}$$

w/ $\check{\mathcal{L}} \longrightarrow \check{\mathcal{X}}_{an}$ v. ample rel. base. (gives

$$\begin{tikzcd} \check{\mathcal{X}} \arrow[r, hook] \arrow[dr] & \mathbb{CP}^N \times \Delta^* \arrow[d] \\ & \Delta^* \end{tikzcd}$$

G)

$\check{\Omega}$ holo. n form, non-deg. on fibers.

$\Rightarrow$ proj. smooth alg. var.

$$\begin{array}{c} \check{\mathfrak{X}} \\ \downarrow \\ \operatorname{Spec} \mathbb{K} \end{array}$$

"Laurent expansion of $\check{\mathfrak{X}}_{an}$".

(comes w/ a projective embedding, ~~oth~~ over $\mathbb{K}$).

Imagine:

X, $\check{\mathfrak{X}}$ mirror pair, certified by some sort of T-duality.

Large cx. str limit assumption on $\check{\mathfrak{X}}$:

monodromy $T : \operatorname{Aut}(H^{*}(\check{X}_q, \mathbb{Q}))$ is maximally unipotent,

e.g. $(T - I)^{n+1} = 0$ & $(T - I)^{n} \neq 0$.

(mirror to $[\omega]^n \neq 0$, so better have this anyway).

Hypothesis: "core HMS"

In this setup, say we're given full sub-categories $\mathcal{A} \subset \mathcal{F}$

& $\mathcal{B} \subset \operatorname{perf} \check{\mathfrak{X}}$ perfect derived category (dg model), s.t. $\mathcal{B}$ split-generates $\operatorname{perf} \check{\mathfrak{X}}$.

and a quasi-equivalence $\mathcal{A} \simeq \mathcal{B}$.

(maybe approachable using tropical methods? known only for a few cases so far).

Generation thm: (P-Sheridan).

Core HMS $\Rightarrow$ $\mathcal{A}$ split-generates $\mathcal{F}$, & hence $\mathcal{F}$ is (homologically) smooth,

& moreover $\mathcal{F} \simeq \operatorname{perf} \check{\mathfrak{X}}$. (HMS holds).

Isomorphism theorem (P.S.)

Core HMS $\Rightarrow$ open/closed maps $HH_*(\mathcal{F}) \xrightarrow{\mathcal{OC}} QH^*(X) \xrightarrow{\mathcal{CO}} HH^*(\mathcal{F})$.

are isomorphisms.

& moreover, $\int_X [D]^{*n} = \int_{\check{\mathfrak{X}}_q} \check{\Omega} \wedge (\nabla_{q\frac{\partial}{\partial q}})^n \check{\Omega}$.

↑ in quantum cohomology, e.g. a series in q. (counts of rat'l curves)

where $\check{\Omega}$ is normalized so that Floer-Poincaré duality in $\mathcal{A}_{\text{core thus}} \longleftrightarrow$ Serre duality in $\check{\mathfrak{X}}$ w.r.t. $\check{\Omega}$ - trivialization of canonical bdle.

"Phase 2" (w. Sheridan & Ganatra):

Explores the role of Gauss-Manin connections / VHS.

Use that to show that core HMS $\Rightarrow$ $\check{\Omega}$ is Hodge-theoretically normalized, & $\check{\mathfrak{X}}$ is in a canonical coordinate.

$$\begin{array}{c} \check{\mathfrak{X}} \\ \downarrow \\ \mathrm{Spec}\,\mathbb{K} \end{array}$$

$$H^n_{DR}(\check{\mathfrak{X}}) = \underset{\uparrow}{\overset{\check{\Omega}\ \in}{W_{2n}}} \supset \cdots \supset W_0 \supset 0$$

monodromy weight filtration

want: $\nabla \check{\Omega} \in W_{2n-2}$: (normalization), &

$(\nabla_{q\frac{\partial}{\partial q}})^2 \check{\Omega} \in W_{2n-4}$. (canonical coordinate).

Expected consequence:

Sheridan proved HMS for quintic 3-fold, up to an undetermined mirror map.

His thm $\Rightarrow$ · the mirror map $\sim$ the std. one

· $\check{\Omega}$ is standardly normalized

· Candelas et al. curve counts follow. (w/o computing those numbers!)

Closed string data:

① − $H^\bullet$ — graded unital $\mathbb{K}$-algebra.

② − $H_\bullet$ — " " $H^\bullet$-module.

③ pairing $H_\bullet \otimes H_\bullet \longrightarrow \mathbb{K}$

④ "marking" $\kappa \in H^2$.

morphisms of such data:

$H^\bullet \xrightarrow{f^\bullet} K^\bullet$ homomorphisms, resp. marking.

$H_\bullet \xrightarrow[f_\bullet]{} \cancel{K}_\bullet$ ← isometry w.r.t. pairing.

Ex: (i) $H^\bullet = HT^\bullet(\check{X}) = \bigoplus_{p+q=\bullet} H^p(\Lambda^q T\check{X})$.

$H_\bullet = H\Omega_\bullet(\check{X}) = \bigoplus_{p-q=\bullet} H^p(\Omega^q_{\check{X}})$.

pairing: $\langle a, b\rangle = \int_{\check{X}} a \wedge b$. maybe sign switch?

Marking θ is the Kodaira-Spencer class in $q\frac{d}{dq} \in H^1(T\check{X}) \subset H^2$.

(ii) $(\mathcal{A}, \mu)$ is a proper A_∞ category, i.e. $H^*(\hom(X_1, X_2))$ finite dim'l.

⇒ closed string data from Hochschild invariants.

$H^\bullet = HH^*(\mathcal{A})$

$H_\bullet = HH_*(\mathcal{A})$ ← Mukai pairing

Marking $\kappa = \left[q \frac{d\mu^\bullet}{dq}\right] \in HH^2(\mathcal{A})$. (need to pick bases to define, but result independent of choice.)

a quasi-equiv. $\mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2 \Rightarrow$ isomorphic closed-string data..
(fix by $\sqrt{Todd}$, to ~~allow~~ make identif. of $HH^\cdot$ & $HH_\cdot$ compatible).

Modified, global HKR isomorphisms give an isomorphism

(Gerstenhaber-Schouten-Kontsevich) $HT^\bullet(\check{\mathcal{X}}) \cong HH^\cdot(\mathrm{perf}\,\check{\mathcal{X}})$

etc. etc.

(not in literature: compatible w/ markings: since Kodaira Spencer class $\mapsto [q \frac{d\check{y}}{dq}]$.

(iii) $H^\cdot = QH^*(X)$

$H_\cdot = QH^{n+\bullet}(X) \qquad \langle a, b \rangle = \int_X a * b$

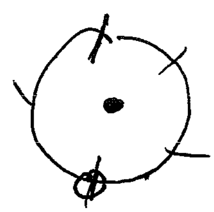
$K = [D]$. (same as classical product ~~in this case~~ b/c $\int_X$)

CO:

$$QH^\cdot(X) \xrightarrow{\mathcal{CO}} HH^\cdot(\mathcal{F})$$

&

$$QH^{n+\bullet}(X) \xleftarrow{\mathcal{OC}} HH_\cdot(\mathcal{F})$$



Prop: $(\mathcal{CO}, \mathcal{OC})$ give a morphism of closed string data.

Least std. part: $\mathcal{OC}$ respects pairing $\langle \mathcal{OC}(a), \mathcal{OC}(b) \rangle$

$$= \langle a, b \rangle_{\mathrm{Muk}}.$$

Maximal unipotent monodromy $\Rightarrow \Theta^n \in H^n(\Lambda^n T\check{\mathcal{X}})$

(Kodaira-Spencer class

K-S class non-zero.. (this is all that's needed)
Here, follows from Hodge theory.

Also: need $\mathcal{F}$ has weak CY str.,

$$\mathcal{F} \xrightarrow{\simeq} \mathcal{F}^\vee[n].$$

+ compatibility w/ $\mathcal{OC}$ maps.

According to Abouzaid's smoothness criterion, ~~to~~ (not in the criterion, actually).

say that $\mathcal{A}$ is ~~be~~ smooth & generates $\mathcal{F}$, suffices to find

$\sigma \in HH_n(\mathcal{A})$ s.t. $\mathcal{OC}(\sigma) = 1 \in QH^*$.

In this context, $\mathcal{OC}$ & $\mathcal{CO}$ are "dual" maps.

Enough to show that $\mathcal{CO}([D]^n) \neq 0$.

generator for QH^{2n}

That follows, since $\mathcal{CO}$, the map on $HH^\cdot$ induced by

$\mathcal{A} \simeq \mathcal{B}$, and HKR respect ring structures & markings.

& fact $\theta^n \neq 0$.

$\Rightarrow$ generation. $\boxtimes$.

$\mathcal{CO}([D]^n) \neq 0$. $\Rightarrow$ $\mathcal{CO}$ is injective.

$a \neq 0 \in QH^*(X)$

$\Rightarrow \exists b$ s.t. $a * b = \omega^n$.

so $\mathcal{CO}(a) * \mathcal{CO}(b) = \mathcal{CO}(\omega^n) \neq 0$.

Dually, $\mathcal{OC}$ is surjective.

$\mathcal{F}$ smooth $\Rightarrow$ Mukai pairing non-degenerate

& $\mathcal{OC}$ is an isometry $\Rightarrow$ $\mathcal{OC}$ ~~isom~~ injective (& Dually, $\mathcal{CO}$ surjective).

Isomorphism.