P. Seidel Talk I : Fibrewise compactification of Lefschetz Fibrations.

(I) $\pi : E \longrightarrow \mathbb{C}$ Lefschetz fibration.

- exact symplectic, $\omega_E = d\theta_E$.
- $c_1(E) = 0$.
- fibre M also has $\omega_M = d\theta_M$, $c_1(M) = 0$.

(II) Fibrewise compactification $\bar{\pi} : \bar{E} \longrightarrow \mathbb{C}$ by adding a smooth divisor at ∞, $\delta E = \bar{E} \setminus E$.

Correspondingly, $\bar{M} = M \cup \delta M$. No singularities at ∞, i.e. $\delta E \cong \mathbb{C} \times \delta M$.

- $c_1(\bar{E}) = 0$. $(\Rightarrow c_1(\bar{M}) = 0)$.
- $[\omega_{\bar{E}}] = PD(\delta E)$.

OR $(\omega_{\bar{E}} - \delta E = d\theta_{\bar{E}})$, "in the sense of currents"

(same on $\bar{M}$).

(I) $\rightsquigarrow$ Fukaya category $\mathcal{F}(\pi)$, $\mathbb{Z}$-graded A_∞-category over $\mathbb{C}$.

The fibrewise compactification gives rise to a deformation

$\mathcal{F}_q(\bar{\pi})$ over $\mathbb{C}[[q]]$. More generally, given

$\boxed{b \in H^2(\bar{E}) \otimes q\,\mathbb{C}[[q]]}$ (so leading coeff. 0), so $b = q b_1 + q^2 b_2 + \cdots$

we can define a deformation $\mathcal{F}_{q,b}(\bar{\pi})$. "bulk term."

One can think of it as having a "q-dependent symplectic form"

$$[\omega_{\bar{E},q}] = -\log(q)\,[\delta E] - b(q).$$

Count holomorphic curves u with $e^{-\int_u \omega_{\bar{E},q}} = e^{\log(q)\, u \cdot \delta E} \cdot e^{\int_u b(q)}$

$$= q^{u \cdot \delta E} \cdot e^{\int_u b(q)}$$ invertible formal power series.

Example:

Suppose $b = \beta(q)\,[\delta E]$, $\beta \in q\,\mathbb{C}[[q]]$.

Then,

$$q^{u\cdot\delta E}\cdot e^{\beta(q)\,u\cdot\delta E} = \left(q e^{\beta(q)}\right)^{u\cdot\delta E}.$$

This corresponds to changing $q \longmapsto q e^{\beta(q)}$.

More generally, reparametrization acts by

$$b(q) \longmapsto b\left(q e^{\beta(q)}\right) + \beta(q)\,[\delta E].$$

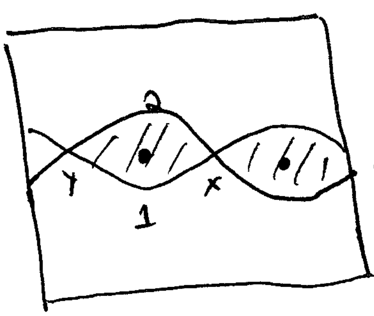
A basis of Lefschetz thimbles gives rise to a full subcategory $\mathcal{A} \subseteq \mathcal{F}(\pi)$ $\left(\mathcal{A}_q \subset \mathcal{F}_q(\bar\pi),\ \& \ \mathcal{A}_{q,b} \subset \mathcal{F}_{q,b}(\bar\pi)\right)$;

the inclusion is a derived equivalence. (Known, but not in literature!)

$\Rightarrow$ deformation theories are the same.

$\mathcal{A}$ (and the other versions) can be computed inside a fibre.

Example: This is the fiber:



$M \subseteq \bar M$ 2-punctured torus. torus.

$\hom_{\mathcal{A}}(V_1, V_2) = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y$.

$\circlearrowleft$ $\mu^1_{\mathcal{A}}$ only A_∞ str.

$= 0$ (no strips b/c of punctures).

$\mathrm{Ob}(\mathcal{A}) = \{V_1, V_2\}$

$$\hom_{\mathcal{A}}(V_i, V_j) = \begin{cases} CF^*(V_i, V_j) & i<j \\ \mathbb{C}\cdot e_{V_i} & i=j \\ 0 & i>j. \end{cases}$$

$\hom_{\mathcal{A}_q}(V_1, V_2) = \mathbb{C}[[q]]x \oplus \mathbb{C}[[q]]y$.

$\mu^1_{\mathcal{A}_q}(x) = qy - qy = 0$.

So $\mathcal{A}_q$ is a trivial deformation $\leadsto$ "(one (x)":

(explicitly; vanishing cycles become isotopic, so $\exists$ a 2-sphere. Only visible in q deformation!).

Let $A, B \in H^2(\bar{E})$ be the classes of the two components of $\delta\bar{E}$. If we set

$$b = \alpha(q)A + \beta(q)B.$$

Then, hol. curves in $\bar{M}$ should be counted w/ additional weights

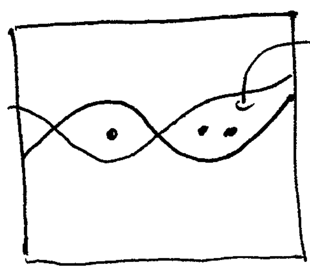
$$\left(e^{\alpha(q)}\right)^m \qquad \left(e^{\beta(q)}\right)^n \qquad (m, n \geq 0 \text{ the \# of times passing through 2 pts.})$$

$$\mu^1_{A_{q,b}}(x) = \left(q e^{\alpha(q)} - q e^{\beta(q)}\right) y$$

Generally a non-trivial deformation.

Rmk:

If instead

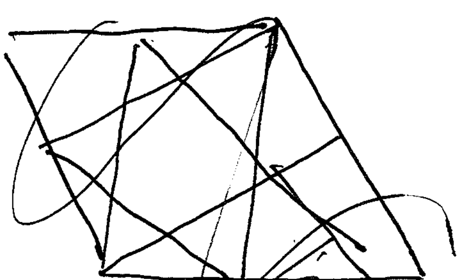


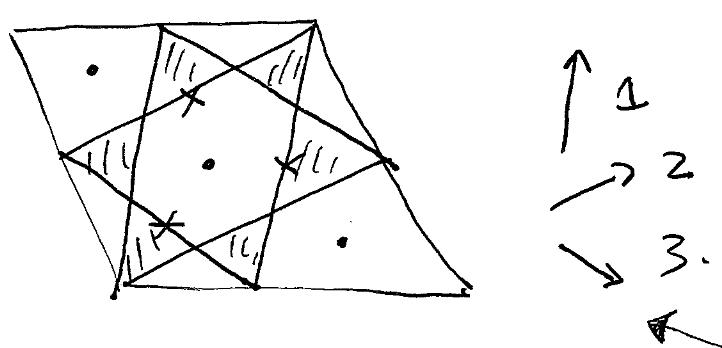
2 pts.

Then, $\mu^1_{A_q}(x) = (q - q^2) y \neq 0$.

(remains non-zero for all choices of b!).

Ex. 2: for the Toric mirrors of $\mathbb{CP}^2$,





the interesting ~~mirror~~ datum is

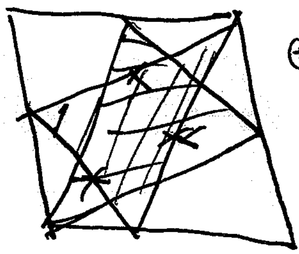
$$\mu^2 : CF^*(V_2, V_3) \otimes CF^*(V_1, V_2) \longrightarrow CF^*(V_1, V_3); \text{ 6 triangles above}$$

(vanishing cycles supposed to carry line bdls. of holonomy -1; e.g. non-trivial spin structures. Mark pt. on L; if triangle passes through x, count w/ -1, else $+1$).

Compactify:

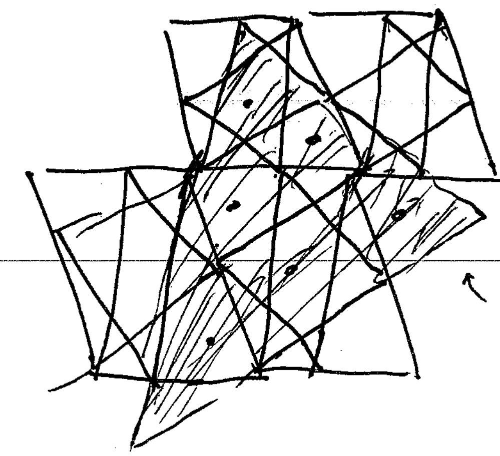
Two big triangles, same endpoints? three × pts.

First:



$$(+1)\,q + (-1)(-1)(-1)\,q = 0.$$

More triangles:



↑ q^6 type term.

$$\eta^2_{\mathcal{A}_q} = \eta^2_{\mathcal{A}} \cdot \left(1 - q^3 - q^6 + q^{15} + \cdots\right)$$

The deformation $\mathcal{A}_q$ is "trivial". ↑ (differs only by change of coordinates!)
(sometimes need non-triv. A_∞ automorphism)

<u>Remark</u>: If we take a non-trivial $b \in H^2_{cpct}(E) \otimes q\mathbb{C}[[q]]$

$$\cong q\,\mathbb{C}[[q]],$$

yields a non-trivial $\mathcal{A}_{q,b}$ (mirror to a non-commutative deformation of $\mathbb{P}^2$.)

<u>Rmk</u>: $E \cong (\mathbb{C}^\times)^2$ contains an exact Lag'n torus T^2 which survives into $\mathcal{F}_q(\bar{\pi})$. Using this and the fact that

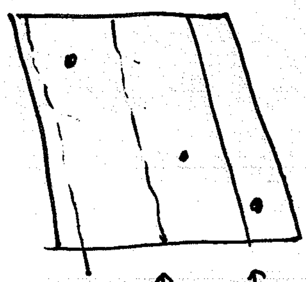
$$HH^2(\mathbb{CP}^2, \mathbb{CP}^2) = H^0(\mathbb{CP}^2, \Lambda^2 T\mathbb{CP}^2),$$

(so only deformations are non-commutative <u>ones</u>),

one can show a priori that $\mathcal{F}_q(\bar{\pi})$ is trivial.

(If b non-zero, integrates non-trivially over T^2, so argument no longer applies.)

Let's take the previous example and add 9 more vanishing cycles:



4	5	6
7	8	9
10	11	12.

coincident families.

The outcome is the Lefschetz fibration coming from an anticanonical Lefschetz pencil on a cubic surface.

In this case, $\mathcal{F}_q(\pi)$ is a trivial deformation of $\mathcal{F}(\pi)$. **Does not** follow from abstract deformation theory.

Explanation: (using MS; assume $q \in \mathbb{C}$ for now):

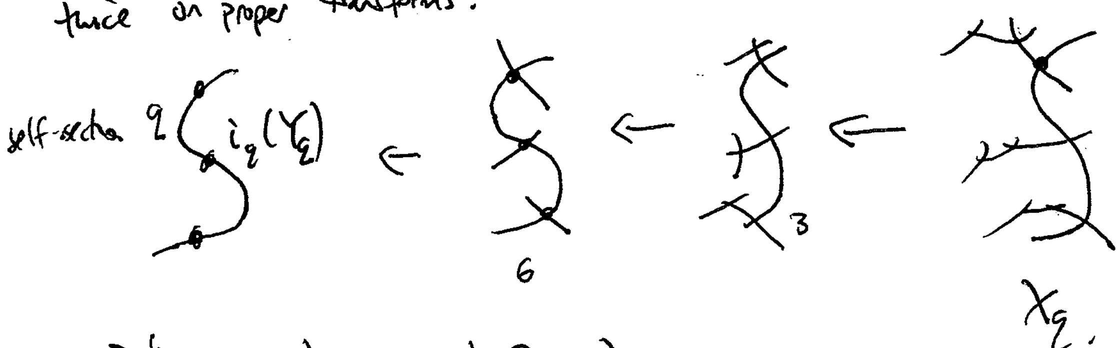
Consider the elliptic curve $Y_q = \mathbb{C} / \mathbb{Z} \oplus \frac{\log(q)}{2\pi i}\mathbb{Z}$

$$Z_q = \left\{0, \tfrac{1}{3}, \tfrac{2}{3}\right\} \subset Y_q \quad (\mathbb{Z}_3 \text{ subgp.}).$$

$F_q = \mathcal{O}(Z_q)$ (deg. 3 line bundle) yields an embedding

$$Y_q \xhookrightarrow{i_q} \mathbb{P}(H^0(F_q)^\vee) \cong \mathbb{P}^2.$$

Take $\mathbb{P}^2$, blow up $i_q(Z_q)$, and then carry out the same blow up twice on proper transforms.

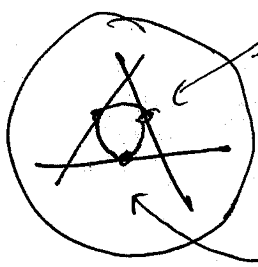


$\mathbb{CP}^2$ blown up 9 times; has an exceptional collection; can see how they restrict to elliptic curve; they're exactly mirrors to our vanishing cycles; from [illegible] fibers/ total spaces [illegible]/etc.),

$$D^b Coh(X_q) \cong D^b \mathcal{F}_q(\pi).$$

Actually, X_q is independent of q. (even though elliptic curves change!)

$\mathbb{CP}^2$



elliptic curve doubly tangent.

$i_q(Z_q)$ are collinear points (by def'n, $\exists$ a section that vanishes at pts.)

So always blowing up same pts, which are three collinear.

& proper transform & blow up same as taking three lines & blowup, b/c ell. curve doubly tangent.

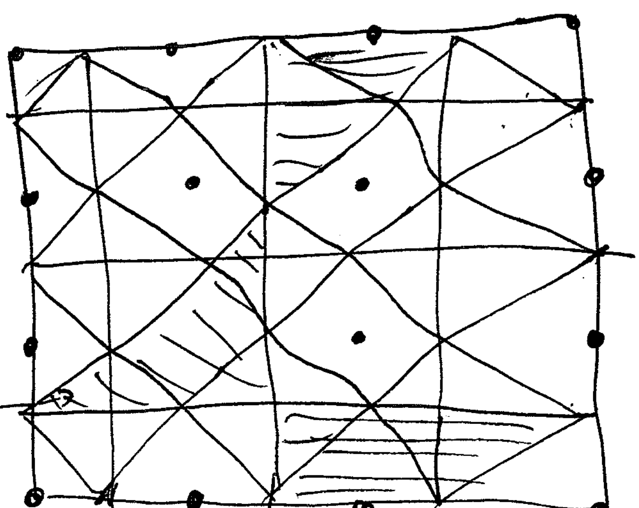
~~to~~ So independent of elliptic curve!

Last example:

Consider the Lefschetz fibration obtained from an anticanonical Lefschetz pencil (of cubics) on ~~$\mathbb{P}^2$~~ $\mathbb{F}_1$

^ Hirzebruch surface, blow up at a point. (so forget one of the punctures)

Here, the deformation $\mathcal{F}_q(\bar{\pi})$ is non-trivial, but there is a choice of b such that $\mathcal{F}_{q,b}(\bar{\pi})$ is trivial.



(Geometry of Lefschetz fibrations singles out a ray in moduli space of deformations s.th. the category is trivial).

the culprit for $\mathbb{F}_1$ (b/c lost a puncture; this would have cancelled in deform's of $\mathbb{P}^2$?).

$$\eta^2 = \underset{\substack{\uparrow ? \\ (1+q)}}{-i} + iq + O(q^2)$$