Seidel II :

$\pi : E \longrightarrow \mathbb{C}$ Lefschetz fibration

$\mathcal{A} \subseteq \mathcal{F}(\pi)$ Fukaya category

$\bar{\pi} : \bar{E} \longrightarrow \mathbb{C}$ fibrewise compactification,

together with $b \in H^2(\bar{E}) \otimes q\,\mathbb{C}[[q]]$.

$\mathcal{A}_q \subseteq \mathcal{F}_q(\bar{\pi})$ $(b=0)$

$\mathcal{A}_{q,b} \subseteq \mathcal{F}_{q,b}(\pi)$ (general b).

Lemma: $\mathcal{A}_{q,b}$ is trivial iff $\left[\frac{\partial}{\partial q} \mu_{q,b}\right] \in HH^2(\mathcal{A}_{q,b}, \mathcal{A}_{q,b})$ vanishes.

(Closed-open, actually).
Open-closed string map

$$CO_{q,b} : H^*(E)[[q]] \longrightarrow HH^*(\mathcal{A}_{q,b}, \mathcal{A}_{q,b})$$

General feature: $CO_q(-[\omega]) = CO_q(-[\delta E])$

$$= \cancel{\left[q \frac{\partial}{\partial q} \mu_q\right]} \cdot \left[q \frac{\partial}{\partial q} \mu_q\right].$$

Slightly improved and generalized: P.D. to cycles supported at divisor at ∞:

$$CO_{q,b} : q^{-1} H^*(\bar{E}, E) \oplus H^*(\bar{E})[[q]]$$

$$\downarrow$$

$$HH^*(\mathcal{A}_{q,b}, \mathcal{A}_{q,b}).$$

Want image to be zero, but preimage is almost never 0. (therefore deformation usually non-zero.)

$$CO_{q,b}(q^{-1}[\delta E] + \partial_q b) = [\partial_q \mu_{q,b}] \quad (= CO_{q,b}(-\partial_q [\omega_{\bar{E},q,b}])).$$

$\frac{\partial}{\partial q}$.

Let's consider the simpler case of $\mathcal{A}$.

$$\begin{tikzcd}
H^*(E) \arrow[r] \arrow[d, "\mathcal{CO}"] & HF^*(\rho) \arrow[d] \\
HH^*(\mathcal{A},\mathcal{A}) \arrow[r, "\cong"] & HH^*(\mathcal{A},\mathcal{R})
\end{tikzcd}$$

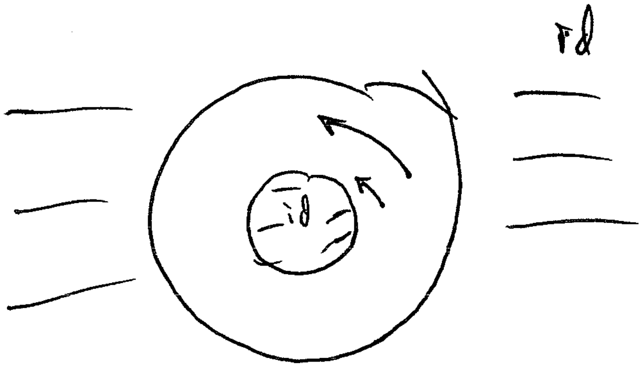
← additional fixed pts.

this iso., whereas, this is not. This is the correct $\mathcal{CO}$ map.

ρ = rotation at ∞, $\mathcal{R}$ = the associated $\mathcal{A}$-bimodule.

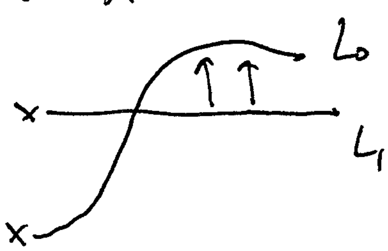
The $\longrightarrow$ arrows are continuation maps. The lower ones comes from a quasi-iso $\mathcal{A} \to \mathcal{R}$.

To construct ρ, take.

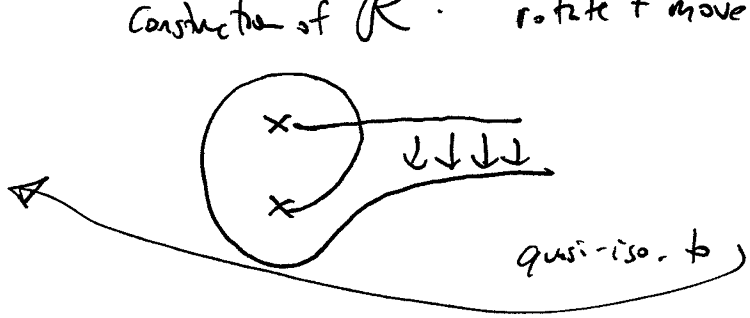


& combine it w/ translation in $-i$ direction.

Morphisms in $\mathcal{A}$:



Construction of $\mathcal{R}$: rotate + move down.



~~The,~~

Correspondingly, we have

some extension fitting in w/ rel. framework (built for symplectic auto. that preserves q divisor, so has q^{-1} terms... somehow??).

b/c TO here always intersects divisor once.

$$\begin{tikzcd}
H^*(\bar{E})_q = q^{-1}H^*(\bar{E}, E) \oplus H^*(\bar{E})[[q]] \arrow[r] \arrow[d, "\mathcal{CO}_{q,b}"] & HF^*(\bar{\rho}) \arrow[d] \\
HH^*(\mathcal{A}_{q,b}, \mathcal{A}_{q,b}) \arrow[r, "\cong"] & HA^*(\mathcal{A}_{q,b}, \mathcal{R}_{q,b})
\end{tikzcd}$$

always (b/c map of bimodules, reduction to $q=0$ is a quasi-iso.).

From now on, assume that our Lefschetz fibration comes from an anticanonical Lefschetz pencil. (& blow-up base locus; (canonical extension over $\mathbb{CP}^1$)).

It then comes w/ a preferred cpctification

$$\begin{tikzcd}
E \arrow[d, "\pi"] & \subset & \bar{E} \arrow[d, "\bar{\pi}"] & \subset & \bar{E}| \arrow[d, "\bar{\pi}|"] \\
\mathbb{C} & = & \mathbb{C} & \subset & \mathbb{CP}^1
\end{tikzcd}$$

"sideways cpctify"

Assume also that b extends "sideways" to $b| \in H^2(\bar{E}|)_q$. (same thing w/ q^{-1} term).

Then, the continuation map fits into a long-exact sequence

b/c monodromy is id, so cpctified fiber.

$$\cdots \longrightarrow H^{*-2}(\bar{M})[[q]] \xrightarrow[S_{q,b|}]{} H^*(\bar{E})_q \longrightarrow HF^*(\bar{\rho}) \longrightarrow \cdots$$

(If want to know if an elt. here from $H^*(\bar{E})_q$ dies, see whether comes from $H^{*-2}(\bar{M})$).

We have to determine $S_{q,\bar{b}}(1)$. (b/c interested in $HF^2(\bar{\rho})$, & $H^0(\bar{M})$ is $\mathbb{K}\cdot\langle 1\rangle$).

Lemma: $S_{q,b|}(\mathbb{1})$ is the restriction to $\overline{E}$

of

$$s_{q,b|} = \sum_{A} e^{b|\cdot A}\, q^{\delta E|\cdot A}\, Z_A \in H^2(\overline{E|})$$

Z_A = cycle swept out by hol. sections of $\overline{E|} \to \mathbb{CP}^1$ in class A.

Two kinds of A appear:

- "trivial sections", A corresponds to connected components of δM. (every fiber contains copy of δM).

  These satisfy $A \cdot \delta E| = -1$, and Z_A is the corresp. component of $\delta E| \cong \mathbb{CP}^1 \times \delta M$. ($q^{-1}$ term)

- other sections have $A \cdot \delta E| \geq 0$. (b/c blow-up of pencil, normal bdle is pull back of $\mathcal{O}(-1)$ to it, & can compute.)

so, $s_{q,b|} = q^{-1}[\delta E|] + \cdots$ higher order.

(& $S_{q,b|}(f(q)) = f(q) S_{q,b|}(\mathbb{1})$).

Hence, the deformation $\mathcal{A}_{q,b}$ is trivial if:

$$q^{-1}[\delta E|] + \partial_q b| = e^{\psi(q)} s_{q,b|}$$

in $H^2(\overline{E|})_q$, but mod $[F]$ ← b/c this dies when we remove the sideways compactification.

This is an equation for $(b|\psi)$:

$$b| \in H^2(\overline{E|})_q, \quad \psi \in q\,\mathbb{C}[[q]].$$

(Necessity of b is b/c of non-triv. hol-sections).

(Note: if no non-trivial hol. sections, $b = 0$ and $\psi = 0$ solve eq'n.)

Solve order by order in q : outcome:

Lemma: (*) has a unique solution up to the following symmetries:

$$(b|, \psi) \longmapsto \left(b|(q) + \alpha(q)[\bar{M}],\ \psi(q) - \alpha(q)\right)$$

(when take away sideways compactification, this dies anyway).

$$\alpha(q) \in q\mathbb{C}[[q]].$$

& rescaling:

higher terms.

$$(b|, \psi) \longmapsto \left(b|(qe^{\beta(q)}) + \beta(q)[\delta E|],\ \psi(qe^{\beta(q)}) + \cdots\right)$$

$$\beta(q) \in q\mathbb{C}[[q]].$$

Cor: b (restriction of $b|$ to $\bar{E}$) is unique up to "rescaling translations"

(May be zero or non-zero; determined by solving equation).

(Explain tomorrow why we need this direction of b)

(If take b & contract to fiber, get class in H^2(fiber)), & if use that b to define Fuk, it will be much simpler than the general direction.)

Example: Start with an anticanonical Lefschetz pencil on the del Pezzo surface of degree 1. ($\mathbb{CP}^2$ blown up at 8 points). then, $\bar{E}$ is a rat'l elliptic surface ($\mathbb{CP}^2$ blown up at 9 pts.)

Counting sections: if $A\cdot A = 2k-1$ (always odd if potential section), then (A·A = −1, exceptional class, exactly unique exceptional section: 1).

$Z_A = A\cdot z_k$ where the numbers z_k satisfy:

$$\sum_{k=0}^{\infty} z_k q^k = \frac{q^{1/2}}{\Delta(q)^{1/2}} = 1 + 12q + 90q^2 + \cdots$$

"Dirichlet delta fcn."

Let $A_0 \in H^2(\overline{E})$ be the exceptional class (of $\overline{E} \to$ del Pezzo).

Can parametrize the general A as

$$A = A_0 + X - \tfrac{1}{2}(X\cdot X)[\overline{M}] + k[\overline{M}]$$

where X satisfies $\overline{M}\cdot X = 0$, $A_0 \cdot X = 0$.

X's lie in simple (neg-definite), E_8) lattice.

Suppose for simplicity $b1 = 0$. Then,

$$S_q = \sum_A q^{\delta E1\cdot A} \cdot Z_k \cdot A, \quad \delta E = A_0, \& A_0\cdot X = 0$$

$A_0 \cdot A_0 = 1$

$$= \sum_{X,k} q^{\cancel{\delta E}A_0 + \delta \cancel{E}X - \frac{X\cdot X}{2} + k} Z_k (A_0 + X) \qquad \text{mod } [\overline{M}].$$

(Rmk: where's 12? counting "stable sections" e.g. sector glued to -2 curve, Not embedded.).

-2 curves aren't resolv for perturbation that preserves J-hol!!

$$= q^{-1}\frac{q^{1/2}}{\Delta(q)^{1/2}} \sum_X (A_0 + X) q^{-\frac{X\cdot X}{2}}$$

X lying in E_8 lattice.

Massive cancellations: X & $-X$.

$$= q^{-1}\frac{q^{1/2}}{\Delta(q)^{1/2}}\, \Theta_{E_8}(q)\,[\delta E1]$$

q^{-1}

Θ fn associated to E_8 lattice.

Indeed, this is a multiple of $q^{-1}[\delta E1]$. (so $b=0$ works! it will not deform.).

($b=0$ is a sol'n of our eq'n).

(Also the for anticanonical pencil on cubic surface via HMS compute the last time.)
In fact $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$ $b=0$ ok except for $k=1,2$.

If we start with an anticanonical Lefschetz pencil on $\mathbb{F}_1$, we get a non-trivial sol'n.

$$s_{q,b1} = \left(q - \frac{19}{6}q^2 + 2q^3 + \ldots\right) A$$

exceptional class in $\mathbb{F}_1$.

Haven't checked manually that this gives desired higher order terms.
(just solved ODE to first few orders ...).

(should be related via HMS to what one gets for Lefschetz hyperplane thm;
& "flat coordinates" Fano anticanonical
("deformations of pair (X, D) that only move D"..)

~~Next idea~~: