Seidel III

Situation:

$\pi : E \longrightarrow \mathbb{C}$ Lefschetz fibration, coming from an anticanonical Lefschetz pencil.

M fibre

$\bar{\pi} : \bar{E} \longrightarrow \mathbb{C}$ fibrewise compactification.

$\bar{M}$ fibre (symplectic Calabi-Yau).

(we're really interested in this).

$$\begin{array}{ccl} \mathcal{A} & \subseteq \mathcal{F}(\pi) & \text{Fukaya cat. of Lefschetz fibration.} \\ \downarrow & \downarrow & \\ \mathcal{B} & \subseteq \mathcal{F}(M) & \text{Fukaya cat. of the fibre.} \end{array}$$

$\mathcal{A}, \mathcal{B}$ are the full subcategories associated to a chosen basis of Lefschetz thimbles (vanishing cycles). $\mathcal{A}$ generates $\mathcal{F}(\pi)$, and $\mathcal{B}$ split-generates $\mathcal{F}(M)$.

In fact, $\mathcal{A} \to \mathcal{B}$ is an inclusion.

$$\hom_{\mathcal{A}}(V_i, V_j) = \begin{cases} \hom_{\mathcal{B}}(V_i, V_j) & i<j \\ \mathbb{K}e_i & i=j \\ 0 & i>j. \end{cases}$$

(need to choose a strictly unital model for $\mathcal{B}$, technically).

by FO3, except $\dim_{\mathbb{C}} 2$, where you need cyclic deformation theory.

Fibrewise compactification yields deformations $\mathcal{B}_q$, $\mathcal{B}_{q,b}$. related in same way as before.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\!\mid \qquad \cup\!\!\!\mid$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{A}_q \qquad \mathcal{A}_{q,b}$

Note: These are unobstructed deformations. (can arrange for $\mu^0 = 0$, & this is rather an A_∞ subalg.)

"Theorem" Choose b in such a way that the deformation $\mathcal{A}_{q,b}$ is trivial. (this can always be done essentially uniquely). [last time]

Then, $\mathcal{B}_{q,b}$ is defined over a finitely generated subalgebra of $\mathbb{C}[[q]]$.

(& can say precisely what generators are).

(actual aim: determine explicitly in terms of geometry of Lefschetz fibration).

(From now on, omit b from the notation; assume always chosen this way).

Reminder: Maurer-Cartan deformation theory:

Three versions

(1) $\mathfrak{g}$ dg Lie algebra, look at $\alpha \in q\mathfrak{g}^1[[q]]$ satisfying

$$d\alpha + \tfrac{1}{2}[\alpha,\alpha] = 0 .$$

(2) (No formal param, but suitable filtration)

$$\mathfrak{g} = F^1\mathfrak{g} \supseteq F^2\mathfrak{g} \supseteq F^3 \supseteq \cdots$$ pro nilpotent (complete decreasing filtration).

$$d(F^i(\mathfrak{g})) \subseteq F^i(\mathfrak{g})$$

$$[F^i\mathfrak{g}, F^j\mathfrak{g}] \subseteq F^{i+j}\mathfrak{g}$$

We look at $\alpha \in (F^1\mathfrak{g})^1$.

(3) $\mathfrak{g}$ is as in (2), but $\alpha \in \mathfrak{g}^1[[q]]$ non-zero const. term ok, b/c of pro nilpotence.

Example for (1):

Deformation theory of A_∞ algebras, e.g. $\mathcal{B}_q$ as a deformation of $\mathcal{B}$.

$$\mathfrak{g} = CC^*(\mathcal{B}, \mathcal{B})[1]$$

$$H^*(\mathfrak{g}) = HH^*(\mathcal{B}, \mathcal{B})[1].$$

If we had a good grip here ↑, can proceed by classifying deformations;
~~if it's finite, "Thm" follows immediately.~~
(unfortunately, don't have grip on general conjecture how to compute it!)

Example for (2):

Given a graded algebra B, classify A_∞ structures that extend the algebra structure. (no q-param; take appropriate filtered version ~~[illegible]~~ of CC (not whole cplx.), filtered using gradings of B) governed by H^*(assoc-graded).

In situation (2):
Ex-Lemma: Suppose that

$$H^*\left(F^k\mathfrak{g} / F^{k+1}\mathfrak{g} \right) = 0 \quad \forall\, k \geq 2$$
$$* = 1, 2.$$

Then, sol'ns of the M-C eq'n in $\mathfrak{g}$ are classified by $H^1(\mathfrak{g}/F^2\mathfrak{g})$. (Subspace is linear & depends on H^1).

Proof (partial): Suppose $\alpha \in \mathfrak{g}^1$ satisfies

$$\beta = d\alpha + \tfrac{1}{2}[\alpha, \alpha] \in F^k\mathfrak{g} \quad (k \geq 2).$$ (satisfies M-C up to error)

Then, $d\beta \in F^{k+1}\mathfrak{g}$, hence we can write

$$\beta - d\gamma \in F^{k+1}\mathfrak{g} \text{ for some } \gamma \in F^k\mathfrak{g}.$$

Applying a γ-gauge transformation yields

an $\tilde{\alpha}$ s.t. $d\tilde{\alpha} + \frac{1}{2}[\tilde{\alpha}, \tilde{\alpha}] \in F^{k+1}\mathfrak{g}$. $\blacksquare$

In situation (3),

Lemma: Under the same vanishing assumptions as before, sol'ns of M-C in $\mathfrak{g}[[q]]$ are classified by $H^1(\mathfrak{g}/F^2\mathfrak{g})[[q]]$.

(same proof).

Moreover, given a class in $H^1(\mathfrak{g}/F^2\mathfrak{g}) \otimes V$, $V \subseteq \mathbb{C}[[q]]$ lin. subspace

Then, the corresp. MC elt. lies in

$$\varprojlim_k \left(\frac{\mathfrak{g}}{F^{k+1}\mathfrak{g}} \otimes V^k \right) \subseteq \mathfrak{g}[[q]]$$

products of $\leq k$ elements of V.

__Application:__ Let A be an A_∞ algebra,

and P an invertible A-bimodule. A __noncommutative__ divisor is

an A_∞ structure on "variety" "line bundle" ("divisor as a dg scheme", add section as differential)

$$\mathcal{B} = A \oplus P[1] \quad \text{such that } A \text{ is a subalgebra of } \mathcal{B},$$

and the induced A-bimodule structure of $\mathcal{B}/A = P$ is the given one.

A_∞ str. on $\mathcal{B}$:

$\underbrace{\mathcal{A}\otimes \cdots \otimes \mathcal{A}}_{d} \longrightarrow \mathcal{A}$ known $(\mu^d_{\mathcal{A}}$ "preserves weights"

$\underbrace{\mathcal{A}\otimes \cdots \otimes \mathcal{A}}_{p} \otimes \mathcal{P} \otimes \underbrace{\cdots \otimes \mathcal{A}}_{q} \longrightarrow \mathcal{P}$ known $(\mu^{p|1|q}_{\mathcal{P}})$ "preserves weights"

Next:

$\mathcal{A}\otimes \cdots \otimes \mathcal{A} \otimes \mathcal{P} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A} \longrightarrow \mathcal{A}$ "increases weights by 1"

$\mathcal{A}$-bimodule map $\mathcal{P} \to \mathcal{A}$

(leading order part of nc. divisor)

("this is the section cutting out the divisor.")

(it's possible to be trivial $\longleftrightarrow$ "graded scheme $X \oplus \mathcal{L}$" trivial nc divisor")

(all such terms increase weights by 1", b/c $\mathcal{A}$ subalg.).

Example: $\mathcal{A} \subseteq \mathcal{B}$ from Fukaya categories (of π and M),

with $\mathcal{P} = \mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1-n]$ (by weak C-Y property). — fiber dimension

Introduce a new grading on $\mathcal{B}$ by "weight": $\mathcal{A}$ has weight 0

$\mathcal{P}$ has weight -1.

Consider Hochschild cochains that strictly increase weight.

This gives a pronilpotent Lie algebra $\mathfrak{g}$ which classifies nc. divisors.

Now, look at $H^j(F^k\mathfrak{g}/F^{k+1}\mathfrak{g}) \longrightarrow H^{j-2k+1}_{\text{bimodules}}(\hom(\mathcal{P}^{\otimes k}, \mathcal{A}))$.

$\to H^{j-2k}(\mathcal{P}^{\otimes k}, \mathcal{A}) \longrightarrow$

$\hom(?)$ (e.g. $\mathcal{P} \to \mathcal{A}$ & $\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P} \to \mathcal{P}$ account for both sides).

$\downarrow$

$\vdots$ (LES!

Lemma: Assume that

Kontsevich: also can use cycle str. to reduce!

$$(*) \quad H^*(\hom(\mathcal{P}^{\otimes k}, \mathcal{A})) = 0 \quad \forall\, k \geq 1 \quad \forall\, * < 0$$

(no homs of negative degrees).

Then, a nc-divisor is completely determined by its leading order part.

Rmk: In the (geometric) situation of anti-canonical Lefschetz pencils,

(*) is always expected to hold.

(there's an OC string map from a fixed pt. Floer coh. to this, spectral seq. for this by McLean, & ~~...~~ doesn't start till pos. degree, & OC should be a iso!)

Deformation theory of nc-divisors:

A_∞-structures on

$$\mathcal{B}_q = (\mathcal{A} \oplus \mathcal{P}[1])[[q]] \quad \text{w/ suitable properties.}$$

($\mathcal{A}$ and $\mathcal{P}$ are NOT being deformed). ($\mathcal{B}$ so cannot have a curvature term, which would have to lie in $\mathcal{P}$).

This is governed by $\mathfrak{g}[[q]]$.

Lemma: Assuming (*), deformations of a nc-divisor are classified by (a subspace of) $H^0(\hom(\mathcal{P}, \mathcal{A}))[[q]]$ ← finite dim'l, easy geometric interpretation

all finite dimensional, not infinite

need to determine leading order term + dependence on q! LINEAR PROBLEM.

Example: $\mathcal{A}_q \subseteq \mathcal{B}_q$

assume trivial, equal to $\mathcal{A} \otimes \mathbb{C}[[q]]$, & associated bimodule doesn't deform (still has C-Y property, which keeps it fixed).