K. Fukaya — joint w/ Evans, Lekili.

Thm: X_1, X_2, X_3

$$\begin{tikzcd}
\mathcal{F}(-X_1 \times X_2) \times \mathcal{F}(-X_2 \times X_3) \arrow[r, "\text{composition}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright \text{ up to homotopy, as diagram of } A_\infty \text{ bifunctors}"] & \mathcal{F}(-X_1 \times X_3) \arrow[d] \\
\mathrm{Func}(\mathcal{F}(X_1), \mathcal{F}(X_2)) \times \mathrm{Func}(\mathcal{F}(X_2), \mathcal{F}(X_3)) \arrow[r] & \mathrm{Func}(\mathcal{F}(X_1), \mathcal{F}(X_3))
\end{tikzcd}$$

X symp. mfold

$\mathcal{F}(X)$ obj. (L, b, σ)

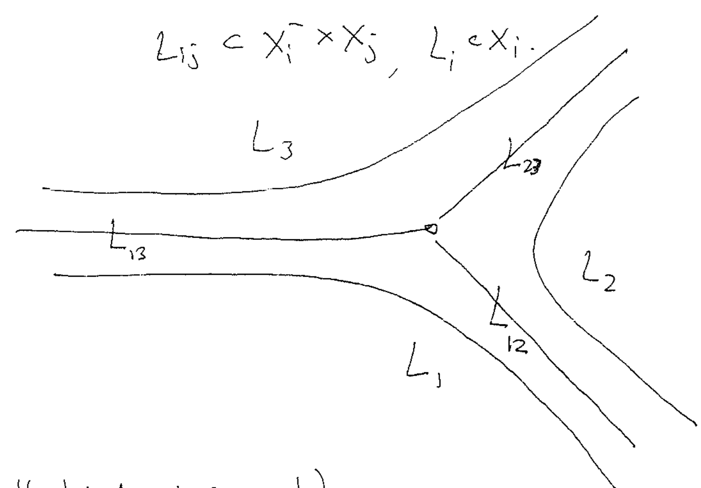
L immersed Lag'n submfld $\tilde{L} \xrightarrow{i} X$ $i^* \omega = 0$.

σ Spin str. (wLOG for this talk) "unobstructed wen, [illegible] category

b bounding co-chain.

$L_{ij} \subset X_i^- \times X_j$, $L_i \subset X_i$.

Main idea:



Y diagram [Lekili - Lipyanskiy]

(commutativity exactly boils down to diagram!)

Say $\tilde{L}$ immersed:

Def: $CF^*(L) = \Omega^*(\tilde{L} \times_X \tilde{L}) \otimes \Lambda_0$. $\{\sum a_i T^{\lambda_i}\}$ Λ_+: all positive.

Ex: L ∞ In this case: $\tilde{L} \times_X \tilde{L} = S^1 \cup 2$ points.

Akaho-~~[illegible]~~ Joyce: (generalizing FOOO):

Have $m_k : CF^*(L)^{\otimes k} \to CF^*(L)$ A_∞ str. (in general curved).

& $b \in CF(L) \otimes \Lambda_+$ satisfying MC eq'n $\sum_k m_k(b, \dots, b) = 0$.

Explain: How to cook up various arrows, & how to prove commutativity.

Previous works:

- Wehrheim–Woodward (+ Ma'u) in case all L's monotone and embedded.
- Lekili–Lipyanskiy: same case, another proof (using Y-diagrams)
- Bottman–Wehrheim: attempt to remove conditions.

Why need to remove embedded? In general when composing, cannot avoid immersed category.

(1) $F(-X_1 \times X_2) \times F(-X_2 \times X_3) \longrightarrow F(-X_1 \times X_3)$

given $L_{12} \subset -X_1 \times X_2$, b_{12}; $L_{23} \subset -X_2 \times X_3$, b_{23} $\rightsquigarrow$ $L_{12} \times_{X_2} L_{23} = L_{13}$

usual Weinstein composition

non-trivial point $\rightsquigarrow$ how to establish b_{13}? & how to get A_∞ functor?

Step 1: Suppose $\mathcal{C}_i$ A_∞ category. then, have Yoneda embedding $\mathcal{C} \longrightarrow Func(\mathcal{C}^{op}, Ch) = \mathcal{C}\text{-mod}$.
(fully faithful!)

So, first, construct a bifunctor

$$\widetilde{Comp}: \begin{matrix} F(-X_1 \times X_2) \\ \times \\ F(-X_2 \times X_3) \end{matrix} \longrightarrow Func(\underbrace{F(-X_3 \times X_1)^{op}}_{= F(-X_1 \times X_3)^{op}}, Ch)$$

$\Longleftrightarrow$ a trifunctor

$$\begin{matrix} F(-X_1, X_2) \\ \times \\ F(-X_2 \times X_3) \\ \times \\ F(-X_3 \times X_1) \end{matrix} \longrightarrow Ch.$$

Step 2: Say have

$\mathcal{L}_{12} = (L_{12}, b_{12}) \in \text{ob } \mathcal{F}(-X_1 \times X_2)$

$\mathcal{L}_{23} \quad (L_{23}, b_{23}) \in \text{ob } \mathcal{F}(-X_2 \times X_3)$.

Then, show

$$\mathrm{Comp}(\mathcal{L}_{12}, \mathcal{L}_{23}) \in \mathrm{Func}(-X_1 \times X_3, \mathrm{Ch})$$

is representable by (L_{13}, b_{13}) where L_{13} is $L_{12} \times_{X_2} L_{23}$, and b_{13} some (!) MC elt. which is shown to exist.

Thus, if representable, get:

$$\begin{tikzcd}
 & & \mathcal{F}(-X_1 \times X_3) \arrow[d, "\text{Yoneda}"] \\
 & & \mathrm{Y_{on}}((\mathcal{F}(-X_1 \times X_3))^{op}) \arrow[u, bend left, "Y^{-1}"] \\
 & & \cap \\
\mathcal{F}(-X_1 \times X_2) \times \mathcal{F}(-X_2 \times X_3) \arrow[rr, "\mathrm{comp}"] \arrow[uuurr, dashed] & & \mathrm{Func}(\mathcal{F}(-X_1 \times X_3)^{op}, \mathrm{Ch})
\end{tikzcd}$$

Omit step 2 for this talk (see last year's talks! homological alg. lemma, says b_{13} comes for free nearly).

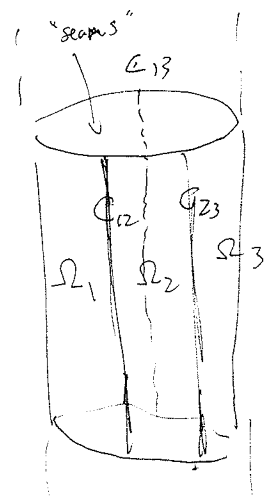
Back to step 1: trifunctor:

Given $L_{12} \subset -X_1 \times X_2$
$L_{23} \subset -X_2 \times X_3$
$L_{31} \subset -X_3 \times X_1$

$\Longrightarrow CF'(L_{12}, L_{23}, L_{31})$ ch. cplx :

by counting

particular case of quilt [WW]



$u_i : \Omega_i \to X_i$ J-hol.

w/ $(u_i|_{C_{ij}}, u_j|_{C_{ij}})$ is a map to $L_{ij} \subset X_{ij}$

$X_{ij} = X_i^- \times X_j$.

$CF(L_{12}, L_{23}, L_{31})$ as Λ_0-module is ~~exact~~ as ch. cplx

$$\underbrace{\Omega^*\left(L_{12} \times_{X_2} L_{23} \times_{X_3} L_{31}\right)}_{\times_{X_1}} \otimes \Lambda_0 .$$ "cycle fiber product")

Differential counts u like just mentioned.

(but of course d^2 may not be zero! Can be corrected if each L_{ij} has bounding co-chains)

Gives a bimodule/tri-functor.

The vertical lines are of the form

$$F(-X_2 \times X_3) \longrightarrow \mathrm{Func}(F(X_2), F(X_3))).$$

This is just a special case when $X_1 = pt.$ of above composition (1).

$\longleftrightarrow$ bifunctor $F(-X_2 \times X_3) \times F(X_2) \longrightarrow F(X_3)$.

The lower horiz. line is purely algebraic.

Alg:

Suppose $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ A_∞ categories.

Then, want $\underset{a}{\mathrm{Func}(C_1, C_2)} \times \underset{b}{\mathrm{Func}(C_2, C_3)} \dashrightarrow \mathrm{Func}(C_1, C_3)$

For functors, this is fairly easy. but on morphisms? Given k nat. trans. on a, & l nat. trans. on b, produce composed nat. trans.

Somewhat hard.

~~Example~~. [Lyubashenko].

Resolve issues like this.

$[\mathcal{C}-\mathcal{D}]$ = cat. of bimodules $\mathcal{C}$-$\mathcal{D}$.

$$\begin{tikzcd}
\mathrm{Func}(C_1, \mathcal{C}_2) \times \mathrm{Func}(C_2, C_3) \arrow[rr, dashed, "?"] \arrow[d, hook, "\text{full faithful}"] & & \mathrm{Func}(\mathcal{C}_1, \mathcal{C}_3) \arrow[d, hook, "\text{full faithful, so can invert.}"] \\
{[\mathcal{C}_1 - \mathcal{C}_2] \times [\mathcal{C}_2 - \mathcal{C}_3]} \arrow[rr, "\text{easy composite}"', "\text{defined using}"] & & \mathcal{C}_1 - \mathcal{C}_3 \text{-bimodule}
\end{tikzcd}$$

$\otimes$ derived tensor product

($\varphi: A \to B$ induces ${}_A B_B$?)

$\otimes$: Given

$$\begin{tikzcd}
\begin{matrix} A \xrightarrow{f} B \\ B \xrightarrow[g]{} C \end{matrix} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft \text{ commutes on objects}"] & A \xrightarrow{g} C \arrow[d] \\
\begin{matrix} {}_A\mathcal{B}_B \\ {}_B\mathcal{C}_C \end{matrix} \arrow[r, "{}_A\mathcal{B}_B \otimes_{\mathcal{B}\mathcal{B}} \mathcal{C}_C"'] & {}_A\mathcal{C}_C
\end{tikzcd}$$

In A_∞ case: $\otimes$ derived tensor product is an easy A_∞ functor

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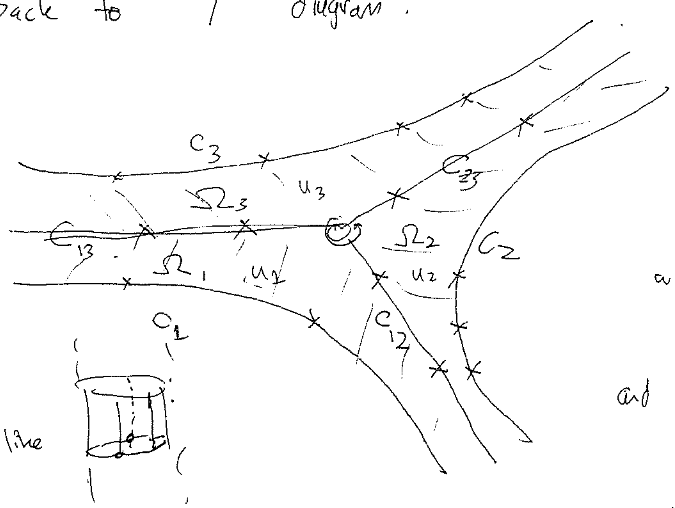
Back to Y diagram:



Consider $u_i : \Omega_i \to X_i$ J-hol.

$(L_i \subset X_i, \quad L_{ij} \subset X_{ij} := X_i^- \times X_j)$

w/ $\operatorname{Im}(u_i|_{c_i}) \subset L_i$

and $\operatorname{Im}(u_i|_{c_{ij}}, u_j|_{c_{ij}}) \subset L_{ij}$

$\circledcirc$ nhood looks like

gives $BCF(L_1) \otimes BCF(L_2) \otimes BCF(L_3)$ $\qquad$ $B :=$ bar cplx.

$\otimes BCF(L_{12}) \otimes BCF(L_{23}) \otimes BCF(L_{13})$

$\otimes CF(L_1, L_{12}, L_2) \otimes CF(L_2, L_{23}, L_3)$

$\otimes \boxed{CF(L_{12}, L_{23}, L_3)} \longrightarrow CF(L_1, L_{13}, L_3)$

corresp. to $\circlearrowleft$.

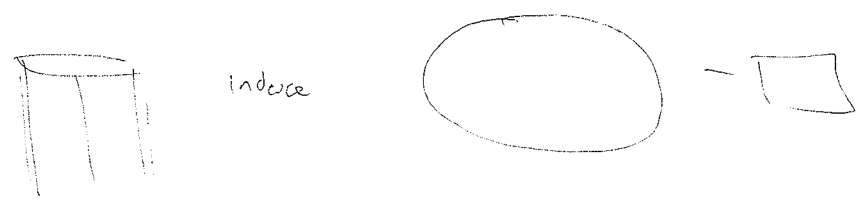
boundary of moduli space $\Rightarrow$ this map is a chain map.

Replace w/ bimodules: get

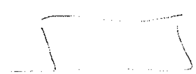
$$\begin{tikzcd} \mathcal{F}(-X_1 \times X_2) \times F_{23} \arrow[r] \arrow[d] & \mathcal{F}(-X_2 \times X_3) \arrow[d] \\ {[F_1, F_2] \times [F_2, F_3]} \arrow[r, "\otimes^{\mathbb{L}}_{\mathcal{F}_2}"'] & {[F_1, F_3]} \end{tikzcd}$$

↑ involves inserting $B\mathcal{F}_2$ (exactly how $BCF^-(L_2)$ appears)

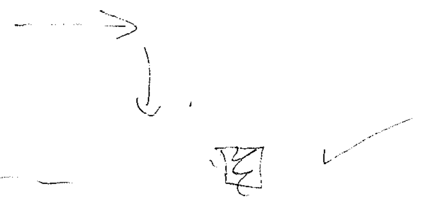
These diagrams



Horiz. composition : is



$BCF(L_{23})$ appears b/c of



& all these terms are exactly ch. homotopy. – – –

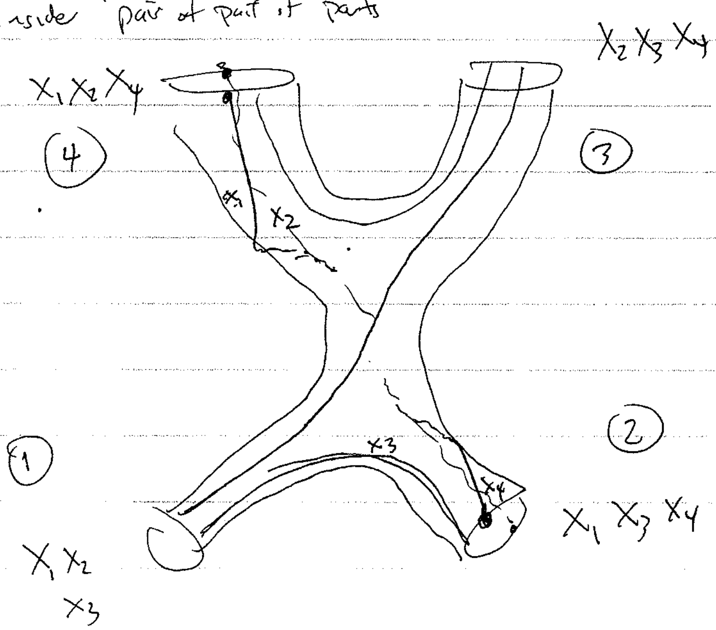
Now, several things are can prove in a related way:

<u>Thm:</u> X_1, X_2, X_3, X_4. Then, want to prove

$$\begin{tikzcd} \prod_{i=1}^{3} F_{i,i+1} \arrow[r, dotted, "③"] \arrow[d, "①"'] \arrow[dr, phantom, "\circlearrowright"] & F_{12} \times F_{34} \arrow[d, "④"] \\ F_{13} \times F_{34} \arrow[r, "②"'] & F_{14} \end{tikzcd}$$

<u>Claim:</u> Associative up to htpy as an A_∞ trifunctor.

Consider "pair of pair of pants"



gives a cobordism btw. different constructions

(fill in —).

Transversality: Consider $L_{12} \subset -X_1 \times X_2$, $L_{23} \subset -X_2 \times X_3$.

Somehow, $L_{13} = L_{12} \times_{X_2} L_{23}$. How to do if this is ~~not~~ transversal?

One way: Take $\varphi_{ij} : X_i^- \times X_j \circlearrowleft$ Hamiltonian.

& use $L_{13}^{\varphi} := \varphi_{12}(L_{12}) \times_{X_2} X_3$. always can be made to work.

& claim: $(L_{13}^{\varphi}, b_{13}^{\varphi}) \sim (L_{13}^{\varphi'}, b_{13}^{\varphi'})$

in $\mathcal{F}(-X_1 \times X_3) \otimes_{\Lambda_0} \Lambda$ — may require neg. energy sol'ns.

bit unsatisfactory; have to invert T.

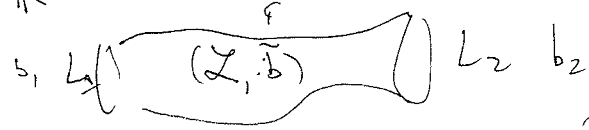
Rmk: no guarantee L_{13}^{φ} & $L_{13}^{\varphi'}$ are diffeomorphic, b/c φ can be quite drastic

If want to work over Λ_0:

Have $(L_{13}^{\varphi}, b_{13}^{\varphi}) \sim (L_{13}^{\varphi'}, b_{13}^{\varphi'})$

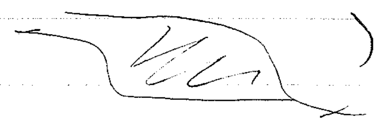
unobstructed Lag'n cobordism.

$X \times T^*\mathbb{R}$



unobstructed, means $\exists$ extension $\tilde{b}$ of b_1, b_2.

[Biran–Cornea]: proves that $\mathcal{L}$ unobstructed doesn't change Floer theory,
(& w/ strong energy bounds for the isotopy, given by perturb. term

Say have (Shelukhin too)? ).

$\mathcal{C}$ A_∞ cat. / Λ_0.
c, c'

Define: $d_{Hofer}(c,c') = \inf\{\varepsilon \mid \star\}$

where $f \in CF^\circ(c,c') \otimes \Lambda_0[T^{-\varepsilon}]$ ~~see~~ $g \in CF^\circ(c',c) \otimes \Lambda[T^{-\varepsilon}]$

~~where~~ s.t. $g \circ f \sim id$
$f \circ g \sim id$

Lemma: Say $\varphi : X \circlearrowleft$ Ham. w/ $d_H(\varphi, id)$

$$\Rightarrow d_{Hof}(\varphi_*(L,b),(L,b)) \leq \varepsilon.$$

Point: Can make ε as small as possible.
For any ε, can use perturbations w/ $d_{Hof} < \varepsilon$.
try to complete $\mathcal{C}$ w.r.t. Hofer distance

Problem: Say $\mathcal{C}$ A_∞ cat / Λ_0.

Want: $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ ← object in completion of $Ob(\mathcal{C})$ w.r.t. d_{Hofer}.
? maybe equiv?

Point: to resolve transversality