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THE ROYAL SWEDISH ACADEMY OF SCIENCES

D. Kaledin, A simple constr. of polynomial Witt vectors 1/27/2015 U. Miami

§1. Motivations: a bit of alternative history

Fix a prime p. For every comm. (assoc.) unital ring A,

we have $W(A) = \varprojlim W_m(A)$ ← truncated Witt vectors
Witt vectors ↗

$W_1(A) = A$

& have $0 \to A \to W_{n+1}(A) \to W_n(A) \to 0$

(start w/ a perfect field, the $W \equiv A$ mod p - but poly. 1 - variable → inf generated but functorial).

Even if $pA = 0$, $W(A)$ usually has no p torsion (like 'char. 0').

First attempts for a "Weil cohomology theory" was made by Serre:

Serre: $H^\cdot(X, W\mathcal{O}_X)$.

didn't work, (b/c we need topological, not coherent, cohomology).

'60s: étale cohomology [Grothendieck] ← coeffs. ℓ-adic $\ell \neq p$.

crystalline cohomology ← indeed sth. p-adic, (uses site, topos, —)

1978: Illusie (+ Deligne):

If X is smooth, finite type / k ← perfect, char. $= p$, then can compute in a way similar to

de Rham Witt complex:

$$W\Omega^\cdot_X = \varprojlim W_m \Omega_X$$

(so comes w/ special filtration!

$W_1 \Omega^\cdot_X = \Omega^\cdot_X$, the usual de Rham cplx., &

$W_{(m)}\Omega^0_X = W_m \mathcal{O}_X$. But in higher degrees some other construction: in particular,

$$gr^i W\Omega^\cdot_X \neq \Omega^\cdot_X \quad i \geq 2.$$

(Q: were there attempts to lift HH. complex to p-adic numbers? these differ from forms in derived category for p.dim X ?).

Nevertheless, have a canonical identification

$$H^\cdot(X, W\Omega^\cdot_X) = H^\cdot_{crys}(X)$$

Zariski topology good enough!

What if the ring is not commutative? Answer is known:

L. Hesselholt 1995 (uses deep theory, top. cyclic hom, don't go into now)

Answer algebraic: we have a functorial abelian group (say $pA = 0$).

$W(A) = \varprojlim W_m(A)$: (A comm. ⇒ Witt vectors, & a ring)

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& successive quotients

$$0 \to A/[A,A] \to W_{m+1}(A) \to W_m(A) \to 0$$

$HH_0(A)$! = ↑ quotient as a vec. space

& $W_1 = A/[A,A]$. (really extensions of Hochschild homology).

Thm: (HKR, 1962): A smooth, ~~finite type~~ $/k$ ← any ring, then

$$HH_*(A) = \Omega^i_A.$$

1982 – [A. Connes, B. Tsygan]:

$$HC_\cdot(A) \Leftarrow HH_\cdot(A)[u^{-1}] \quad |u| = 2$$

w/ first non-trivial differential

$$B : HH_\cdot(A) \to HH_{\cdot+1}(A).$$

In the HKR situation,

$B \longleftrightarrow$ deRham differential d.

Actually discovered 1963 by G. Rinehart

So, one would expect to have $WHH_\cdot(A)$ for any assoc. algebra A, compatible w/ prev. things (e.g. in degree zero, its $W_m(A)$).

However, $HH_\cdot$ is a functor of two variables. Have A – algebra or ring, and M, an A-bimodule.

Say A, B algebras, M an $A^{op} \otimes B$-mod, N a $B^o \otimes A$-mod (say sufficiently flat, or replace)

$$HH_\cdot(A, M \otimes^L_B N) \underset{\text{canon.}}{\cong} HH_\cdot(B, N \otimes^L_A M)$$

"trace-like" ($\mathrm{tr}(AB) = \mathrm{tr}(BA)$))

So, for $M = A^o \otimes V \otimes A$ (free over $A^o \otimes A$; V vector space$/k$, where A a k-algebra.) $(B = k)$.

easy prop: it's same as certain chain cplxes.

$$HH_\cdot(A, M) \cong HH_\cdot(k, A \otimes V)$$

(so once we know $HH_\cdot(k, -)$, can use this to compute any $HH_\cdot(A, M)$) replace by free A-bimodules

Axiomatization: "trace theory", "trace functor"

(K. Ponto, —)

D. Kaledin

Def: A trace functor $f : \mathcal{C} \to \mathcal{E}$ $\mathcal{C}$ a monoidal category, $\mathcal{E}$ anything, is a functor equipped w/ functorial maps

$$\tau_{M,N} : f(M,N) \xrightarrow{\sim} f(N,M)$$ satisfying compatibilities —

(Rmk: we've forgotten about bimodules here, working w/ vec-spaces by above remark). Turns out this automatically gives extension to a theory for algebras and bimodules by a formal procedure (using above remark).

Example: $\mathcal{C} = k\text{-Vect}$, $\mathcal{E} = k\text{-Vect}$

$n \geq 1$ integer

$$F(V) = (V^{\otimes n})_\sigma$$

$\sigma : V^{\otimes n} \circlearrowleft$ is the order n permutation.

$$\tau_{M,N} = \underset{\text{symmetry isomorphism (tensor products in two directions are same)}}{\tau_{M,N}} \circ (\overset{\text{twist by permutation}}{\sigma \otimes \mathrm{id}}) \qquad M^{\otimes n} \otimes N^{\otimes n}$$

Can check this gives a trace functor.

—

So, we need a trace functor (want e.g. W_m + $\mathrm{br\,m\,char.}\ \Theta$, so target can't be $k\text{Vect}$)

$$W_m : k\text{-Vect} \longrightarrow W_m(k)\text{-mod}.$$

Consider $W_m(k)\text{-mod}^{ff}$ free finitely-gen. modules. If k perfect, $W_m(k) \bmod p = k$, so since.

So, Have functor $q : W_m(k)\text{-mod}^{ff} \longrightarrow k\text{-Vect}^f$ (ess. surjective)

This functor is not invertible! (point: can't do canonically:

e.g., $q : GL_d(W_m(k)) \longrightarrow GL_d(k)$ does not split if $d \geq 2$.)

—

For any fin.-gen.-free $V \in W_m(k)\text{-mod}^{(ff)}$, denote

$$V_{(m)} = V^{\otimes p^m} \qquad \text{Denote } G_m = \mathbb{Z}/p^m\mathbb{Z}.$$

Then, G_m acts on $V_{(m)}$.

Denote by $Q_m(V) = \check{H}^0(G_m, V_{(m)})$ (0th Tate cohomology, coker of $(V_{(m)})_{G_m} \to V_{(m)}^{G_m}$)

Remark: Q_m has a natural trace functor structure.

Thm: $Q_m(V)$ only depends on V/p.
(or, more precisely, there exists a functor
$W_m : k\,\mathrm{Vect} \longrightarrow W_m(k)$-modules (image not free, but self-dual)
s.t. $Q_m(V) \cong W_m(V/p)$.

(uniqueness is obvious, & trace functor structure automatically descends)

$W_2(V)$: two ways:
vector space now

$$0 \to (V^{\otimes p})_{\sigma} \longrightarrow W_2(V) \longrightarrow V \to 0$$

$\nwarrow$ coinvariants

but also,

invariants

$$0 \longrightarrow V \longrightarrow W_2(V) \longrightarrow (V^{\otimes p})^{\sigma} \longrightarrow 0.$$

point is that:

$\check{H}^{\ell}(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \underset{\text{can.}}{\cong} V$. (induces these maps for coinvariants & invariants)

Have $\quad V \quad \underbrace{\phi(V)}_{}\quad V$ with $\overbrace{\phi(V)\quad V}^{(V^{\otimes p})^{\sigma}}$, $\underbrace{V\quad\phi(V)}_{(V^{\otimes p})_{\sigma}}$ & mult. map identifies?

$\phi(V)$: p torsion.

General structure: for m, have $2m-1$ terms instead of these 3 terms

Pf of theorem on objects clear, issue is morphisms:

Say have
$$M, N \in W_m(k)\text{-mod}^{ff}$$
& have $a, a' : M \longrightarrow N$. Need to show that: if $a \equiv a' \pmod p$,

then $Q_m(a) = Q_m(a')$.

(Reduce to dim $M = 1$ (b/c Q_m a ps.-$\otimes$ functor), as follows:

Denote by $A = \mathrm{Hom}_{W_m(k)}(M,N)$

action map $M \otimes A \to N$

$M \xrightarrow{\mathrm{id}\otimes\tilde{a}} M \otimes A \to N \rightsquigarrow Q$

$Q(a) = Id + Q(\tilde{a}) \otimes -$

enough to show that $Q(\tilde{a}) = Q(\tilde{a}')$.

where $\tilde{a} : W_m(k) \to A$.

But, $Q(\tilde{a})$ is the image of $a^{\otimes p^m}$ (invariant class) under the projection

$$H^0(G_m, A_{(m)}) \to \check{H}^0(G_m, A_{(m)})$$

~~Rem~~ Universal situation:

Take $A = W_m(k)[S]$ $\quad S = \{0,1\}$.

$a = c_0$, $a' = c_0 + pc_1$

Note that (*) $(a')^{\otimes p^m} = a^{\otimes p^m} + \sum_{f : 1,\dots,p^m \to S} p^{|f|} c_{f(1)} \otimes c_{f(2)} \otimes \cdots \otimes c_{f(p^m)}$ (*)

$|f| = \sum_i f(i)$. (# times have value 1)

$\Rightarrow |f| \geq 1$. if f not zero.

acted on by cyclic group

$S^{p^m} = \coprod S^{p^m}_{(i)}$ where $S_{(i)}$ is the subset of points w/ stabilizer $p^i\mathbb{Z}/p^m\mathbb{Z}$ in G.

(i=m stratum contributes nothing b/c Tate cohomology acts trivially ~~b~~ on free modules).

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One shows easily that

$$\check{H}\left(G_m, W_m^{(k)}[S^{p^m}]\right) = \bigoplus_{0 < i \leq m-1} W_{m-1}(k)\left[S_{(i)}^{p^m} / G_m\right]$$

(The functions f nowhere points).

If f lies in $S_{(i)}^{p^m}$, (its stabilizer is something in (*) w/ some indices equal (b/c symmetry))

$|f|$ is divisible by p^{n-1}. But $|f| \geq 1$.

$\Rightarrow |f| \geq p^{m-i} > m-i$ for every $p, n-i$.

So, $p^{|f|} = 0$ in $W_{m-i}(k)$, so the corresp. term ~~to~~ in (*) vanishes after projection.

(following suggestion of Voevodsky that this might be true).

Didn't say how to relate W_m for different m's. Similar type of argument.

$\rightsquigarrow$ gives in commutative case a Hochschild type complex.

[Kapranov]:
(generalizes 'Teichmuller representatives' to matrix case)

case $d=1$