M. Kapranov, Perverse Schobers on surfaces and Fukaya categories w/ coefficients

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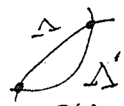
① Motivation: conj. categorical analogues in sense of categorifications

Perverse Schobers	Perverse Sheaves
Triangulated categories $\mathcal{D}$ (or DG)	Vector spaces/k
	$\rightsquigarrow K_0(\mathcal{D}) \otimes_{\mathbb{Z}} k$.
	e.g.
$\mathrm{Fuk}(M^{2n}, \omega)$ $CF(\Lambda, \Lambda')$	$H_n(M^{2n}, k)$
categorifies Lag'n part of H_n - " $\Lambda \cap \Lambda'$.	smooth, cpt, oriented.



?what is an analogue of $\mathcal{F}$?

sheaf-like coefficient data.

$H^n(M^{2n}, \mathcal{F})$
sheaf.

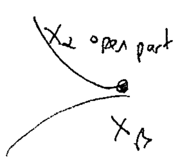
N.B. sheaf theory is local, but Fukaya theory is not local (disks).

Idea [Kontsevich]:

? Local (ignoring disks) categorical approximation
⇣ deformation theory
full blown picture.

② Perverse sheaves. $\mathcal{X} = (X, S)$ — strata which are smooth cplx. submflds
$\mathbb{C}$-mfld. $S = (X_\alpha)$ smooth/$\mathbb{C}$

e.g.



$\mathrm{LocSys}_X \subset \mathrm{Constr}(\mathcal{X}) = \{ \mathcal{F} \mid \mathcal{F}|_{X_\alpha} \text{ local sys.} \}$.

abelian categories

$\cap$

$\mathrm{Perv}(\mathcal{X}) \hookrightarrow D^b_{\mathrm{constr.}}(\mathcal{X})$ (complexes w/ constr. coh.).

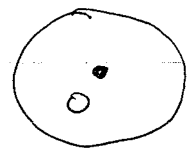
$\mathrm{Perv}(\mathcal{X}) = \{ \mathcal{F}^\bullet \text{ (cplx)} \mid \underline{H}^i(\mathcal{F}) \text{ supp. on codim}_{\mathbb{C}} \geq i$
$+ \text{ dual condition} \}$.

Ex: $\mathcal{F}^\bullet = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ $\mathcal{M}$ hol. $\mathcal{D}_X$-module (even if not regular).

$LocSys_X$, $Constr(\mathfrak{X})$ categorify:
(∞-)stacks of dg-categories.

$Perv(\mathfrak{X})$ not so easily (b/c not so clear what is a cplx. of categories).
"Schober = German for stack"

③ Spherical functors: $\mathfrak{X} = (\mathcal{D}_{|z|<1}, 0)$

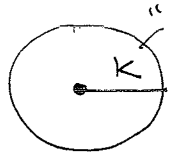


Galligo - Granger - Maisonobe (1982):
$Perv(\mathcal{D}, 0) \sim$ cat. of diagrams

$$\mathcal{F} \qquad \Phi \underset{b}{\overset{a}{\rightleftarrows}} \Psi \quad \text{s.t.} \quad T_\Psi = 1_\Psi - ab$$
$$T_\Phi = 1_\Phi - ba \quad \text{are invertible.}$$

vanishing cycles (for Φ), nearby cycles (for Ψ).

Pf: Choose a "cut"



" $\longleftrightarrow$ 1st instance of choice of Lag'n skeleton"!

$\mathbb{H}^i_K(\mathcal{F}) = 0$ $i \neq 1$, and

$\mathcal{F} \longmapsto \mathbb{H}^1_K(\mathcal{F})$ exact functor of Ab. categories $Perv(\mathcal{D}, 0) \longrightarrow Sh(\mathcal{D})$.
Φ = stalk at 0, Ψ = stalk elsewhere.
$= \mathcal{F}_1$.

a is the generalization map (describes sheaf structure).

There is a way to categorify such data:

Categorification: spherical functor (Anno–Logvinenko).

Say have $\mathcal{D}_0 \underset{f^*}{\overset{f}{\rightleftarrows}} \mathcal{D}_1$ exact functor of (pre)-triangulated cat. (canonical cones)

f^* ^ right adjoint.

~~right~~

So, have: $\mathrm{Cone}\{f \circ f^* \longrightarrow \mathrm{Id}_{\mathcal{D}_1}\} = \underline{T_1}$ twist

$\mathrm{Cone}\{\mathrm{Id}_{\mathcal{D}_0} \longrightarrow f^* \circ f\} = T_0$. cotwist

f called spherical loosely if T_0 & T_1 are auto-equivalences of categories. (⇒ on K_0, induces perverse sheaf on a disc.)

~~⇒ categ~~ $\widehat{\mathrm{Perv}}(\mathcal{D}, 0)$.

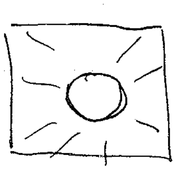
Rmk: vec. space picture: a, b independent, category picture: f, f^* related!

NB: (a) only one f is needed.

(b) Invariantly, have $\underline{f}: \underline{\mathcal{D}_0} \longrightarrow \underline{\mathcal{D}_1}$ a morphism of local systems of categories over S^1 w/ monodromies T_0, T_1.

Every stalk is a spherical functor

spherical morphism.

(c) Consider  $\underline{Y} = \{|z| \geq 1\} \subset \mathbb{C}$.

Then, $\underline{f} \rightsquigarrow$ a sheaf $\mathcal{G}$ of categories on Y

s.t. $\mathcal{G}|_{S^1} = \underline{\mathcal{D}_0}$, $\mathcal{G}|_{Y \setminus S^1} = \underline{\mathcal{D}_1}$, and

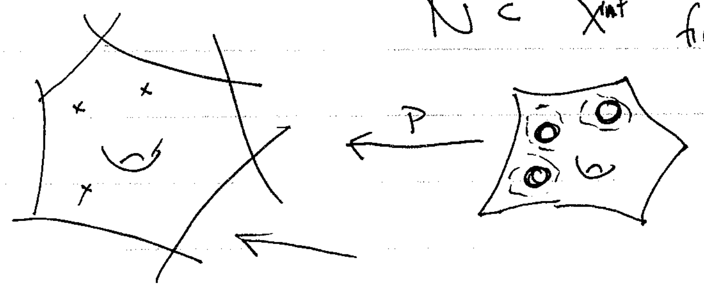
$\underline{f}$ = gluing. (use essentially fact (a); can only prescribe one f this way)

spherical sheaf.

④ Perverse Schobers on surfaces

X C^∞ oriented surface with ∂ and corners.

$N \subset X^{int}$ finite set of "singular points"



$x \in N$

$p \uparrow$

S^1_x circle of directions.

$$\tilde{X} = Bl^{\mathbb{R}}_N(X)$$

Def: A perverse schober on X w/ singularities at N

= a (non-perv.) sheaf $\mathfrak{S}$ of pre-tr. cats. s.t.

- Loc. const. on $\forall\, S^1_x$ & $\tilde{X} - \coprod S^1_x$
- Spherical near $\forall\, S^1_x$.

Rmk: Similar def'n when $X/\mathbb{C}$ any mfld, $N \subset$ divisor w/ normal crossings.

(can still take real blow-up const'n; over simple pt., S^1, over double pt $(S^1)^2$, etc.)

~~cgn~~ (uses std. desc. of Perv on simple coord. normal crossings.)

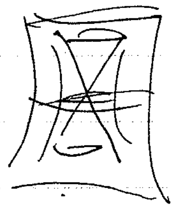
Example: $W : M \longrightarrow X$ is a holom. Lefschetz pencil, proper
M Kähler, X $\mathbb{C}$-curve

Have $m_1, \ldots, m_n \in M$ critical points, and $w_1 \ldots w_n = W(m_i)$ values.

$\rightsquigarrow \mathfrak{S}_W$ perv. Schober on X with singularities in $\{w_1, \ldots, w_n\}$.

Stalk at $x \notin \{w_i\}$ is $\mathrm{Fuk}\, W^{-1}(x)$.

$$\text{"}\Phi\text{" at } w_i = D^b(\mathrm{Vect}/\underset{\text{Novikov}}{\Bbbk}) \xrightarrow[\text{functor}]{\text{sph.}} \mathrm{Fuk}(\underset{\text{nearby}}{W^{-1}(x)})$$



$$\Bbbk \longmapsto \text{vanishing spheres.}$$

Categorifies minimal (Goresky-Macpherson) extension of

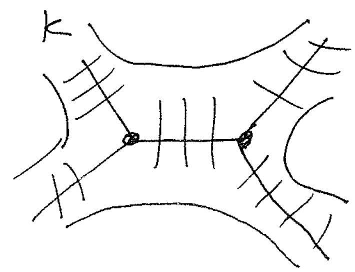
$$R^{\text{middle}} W_* \underline{k}_{M - W^{-1}(\text{crit. values})}.$$

(5) Lagrangian collapse for perv. sheaves and Schobers

(X, ω) exact "Stein" sympl. m'fold.

$\omega = d\alpha \rightarrow$ Liouville field V

Flow$_V$ collapses X to K.



Kontsevich's proposal: $\exists$ intrinsic sheaf of pre-tr. categories $\mathcal{R}_K$ on K such that

$$\mathrm{Fuk}(X) = \text{"}\underbrace{R\Gamma}_{\text{holim}}\text{"}(K, \mathcal{R}_K)$$

holim (in cat. of dg cat's).

Observe: $\mathcal{R}_K$ categorifies $H^{\text{middle}}_K(\mathbb{Z}_X)$ in known examples.

Proposal reformulated: categorify cohomology with support.

Ex. X surface $\supset K$ emb. graph

Easy to see: $\forall\, \mathcal{F} \in \mathrm{Perv}(X)$, $\mathbb{H}^{\neq 1}_K(\mathcal{F}^\bullet) = 0$ (not true for ordinary sheaves, so $\mathcal{F}$ gives right categorification of coh. w/ support).

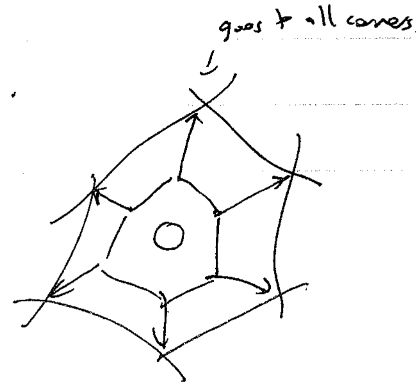
Fix $N=\{x_1,\dots,x_n\}$, K arbitrary graph.

<u>Thm</u> (a): $\exists$ ∞-functor

$$\begin{tikzcd} R_K : \mathrm{Schob}^{2\text{-per}}(X,N) \arrow[r] \arrow[d, "k_0\otimes k"] & \mathrm{Sh}^{dg\,Cat}(K) \arrow[d, "k_0\otimes k"] \\ \mathrm{Perv}(X,N) \arrow[r, "\mathbb{H}^1_K"'] & \mathrm{Sh}^{Vect}(K) \end{tikzcd}$$

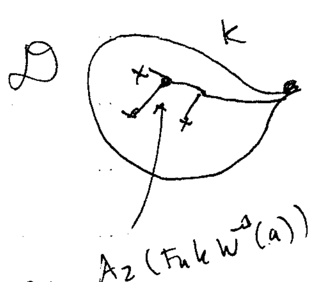
↑ ∞ cat.

(b) Let K be a <u>spanning graph</u> for X containing N.
Then, $R\Gamma(K, R_K(\sigma))$ is coherently independent on the choice of K.



can be denoted
Fuk(X,σ) top. Fukaya category w/ coefficients.

<u>Ex</u>: Let $X =$ disk D with 1 corner (marked pt. on surface),
& $W: M \longrightarrow D$ Lefschetz pencil.



Then $\mathrm{Fuk}(D, \sigma_W) = FS(W)$,
has an exceptional collection by construction.

<u>6 Structure of $R_K(\sigma)$</u>

Waldhausen S-construction (appears in gluing semi-orth. decomp.
[Kuznetsov–Lunts] !!)

$\mathcal{B}$ pre-tr. dg-cat.

$S_n(\mathcal{B})$ = replacement of $A_n(\mathcal{B}) = \{B_1 \to \cdots \to B_n\}$

$\downarrow \partial_0, \ldots, \partial_n$ simplicial: ∂_i, $i \neq 0$ dropping B_i

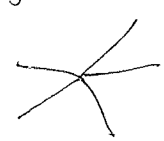
$S_{n-1}(\mathcal{B})$ $\quad \partial_0 = B_2/B_1 \to \cdots \to B_n/B_1$

$\uparrow$ cone of map

Waldhausen: replaces $A_n(\mathcal{B})$ so simplicial identities actually hold (don't quite as stated)

Kontsevich: (w/o coeff.)

$$(R_K)_x = A_n(\mathrm{dg\,Vect}) \quad \text{if} \quad \mathrm{val}(x) = n+1.$$



x

! $C : S_n \to S_n$ s.t. $C^{n+1} = \Sigma^2$ ← shift by 2 well-defined on 2-per. version.
(categorification of eq'n for A_n singularity [Coxeter]!).

Relative S-construction:

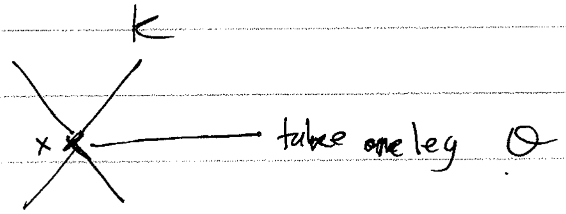
If $f : \mathcal{B} \to \mathcal{C}$, then (made to model fibers of maps in alg. K theory!)

$$\begin{tikzcd} S_n(f) \arrow[r] \arrow[d] & S_{n+1}\mathcal{C} \arrow[d, "\partial_0"] \\ S_n\mathcal{B} \arrow[r, "f_*"] & S_n\mathcal{C} \end{tikzcd}$$

$S_{n+1}\mathcal{C}$ — has semiorth. decomp. $\langle \underbrace{\mathcal{B}, \mathcal{B}, \ldots, \mathcal{B}}_{n}, \mathcal{C} \rangle$

$S_1(f) = \langle \mathcal{B}, \mathcal{C} \rangle$
gluing SOD [Kuznetsov-Lunts].

Our prescription:



look at $f^*_\theta : \Psi_\theta \to \Phi_\phi$ (spherical functor in this direction!)

& $(R_K \sigma)_x = S_n(f^*_\theta)$. Q: why independent of choice of legs?

Thm: Let $f : \mathcal{B} \to \mathcal{C}$ be a ~~spherical~~ functor.

Then, $S_n(f)$ has $n+1$ copies of $\mathcal{B}$

$$\mathcal{B}(0) \quad \mathcal{B}(1) \quad \text{---} \quad \mathcal{B}(n)$$

If f is spherical, then "second orthogonals"

$$\mathcal{B}(0)^{\perp\perp} = \mathcal{B}(1), \quad \mathcal{B}(1)^{\perp\perp} = \mathcal{B}(2), \quad \mathcal{B}(2)^{\perp\perp} = \mathcal{B}(3) \text{ ---}$$

periodicity.

So, $2(n+1)$ periodicity of orthogonals.

In particular, for $n=1$, $S_1(f) = \mathcal{B} \ltimes_f \mathcal{C}$ (Kuznetsov-Lunts gluing).

D. Halpern-Leistner, ~~?~~ I. Shipman '13:

f is spherical $\iff \mathcal{B}^{\perp\perp\perp\perp} = \mathcal{B}, \quad \mathcal{C}^{\perp\perp\perp\perp} = \mathcal{C}$