M. Kontsevich, Multiplication kernel.

[A. Weinstein]: {sympl. manifolds} as a $\otimes$ category

$(X_1,\omega_1)\times(X_2,\omega_2)$ $\otimes$

& $\hom((X_1,\omega_1),(X_2,\omega_2)) = \{\text{Lagr. } L\subset (X_1,-\omega_1)\times(X_2,\omega_2)\}$

(morally a $\otimes$ category, but composition issues)

(Reasonable kind of) Quantization $(X,\omega) \rightsquigarrow \mathcal{H}_X$ Hilbert space (morally fns. on 1/2 the variables)

$L\subset(X,\omega) \rightsquigarrow$ vector space

"$i\hbar$" $(X,-\omega) \longrightarrow \mathcal{H}^*_{(X,\omega)} \cong \overline{\mathcal{H}}_{(X,\omega)}$

imaginary Planck constant

different quantizations

Different quantization:

$(X,\omega) \rightsquigarrow \mathrm{Fuk}(X,\omega)$ A_∞ category.

"$0<\hbar\ll 1$" real Planck constant.

Lag'n correspondences $\rightsquigarrow$ functors

Slogan Integrable system (X,ω) w/ Lag'n section s.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ B \arrow[u, bend left, "s"] \end{tikzcd}$$

$\longleftrightarrow$ comm. assoc. alg. A in Weinstein category

$1 \xrightarrow{\text{unit}} A$ $s(B)$

$A\otimes A \xrightarrow{\text{mult.}} A$ given by

$L\subset(X_1,-\omega)\times(X_2,-\omega)\times(X,\omega)$

$= \{(x_1,x_2,x_3) \mid \pi(x_1)=\pi(x_2)=\pi(x_3)=b$, and $x_1+x_2=x_3$ in $\pi^{-1}(b)$ using addition law given by section $s\}$.

(Can apply these functors to integrable system in both cases) - Here, get:

"comm. assoc. alg. in $\otimes$ cat. of A_∞ cat" $\leftrightsquigarrow$ (derived) comm. alg. geometry!

automatically get more:

"counit or trace": $t: A \to 1$ such that

$$A \otimes A \xrightarrow{\text{mult.}} A \xrightarrow{t} 1 \quad \text{is non-degenerate}$$

$$(,)^{-1} : 1 \to A \otimes A.$$

Non-degenerate means:

$$\begin{tikzcd} A \arrow[r, "id_A \otimes (-,-)^{-1}"] \arrow[rr, bend right, "= id_A"'] & A \otimes A \otimes A \arrow[r, "(-,-) \otimes id_A"] & A \end{tikzcd}$$

For instance, the pairing on $(X, \omega) \times (X, \omega)$ is $\{(x_1, x_2) \mid x_1 + x_2 = 0 \text{ in } \pi^{-1}(b)\}$

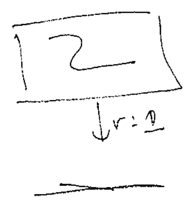
Goal: explain this more in algebraic situation.

From now on, assume $X = T^*M$.

Basic example: Hitchin system. $\mathcal{M} = \{$moduli of rank r bdles on curve $C\}$,

$T^*\mathcal{M}$ Higgs bundles

$\downarrow$

$B = \{ \Sigma \subset T^*C$ spectral curve  $\}$.

In this case, what's the quantization functor?

(e.g., for cpx. sympl. mfld, Fukaya category should be $\longleftrightarrow$ D-modules)

Ans: D_M-modules $\supseteq$ Holonomic D-modules

integrable system $\xrightarrow{?}$ holonomic D-module on M^3 "multiplication kernel."

Why should it exist? gives str. of a $\otimes$ category on holonomic D_M-modules

Geom. Langlands roughly says

$$D\text{-mod}(M^3) \simeq QCoh(N^3) \supset \mathcal{O}_{\text{triple diagonal}}$$

$M = Bun_G$ ($\uparrow$ bundles on G); $N = LocSys_{G^\vee}$ $\leftarrow$ bundles on $G^\vee$ w/ connection.

& $\mathcal{O}_{\text{triple diagonal}}$ should go to multiplication kernel.

Also, QCoh has a $\otimes$ $\longleftrightarrow$ gives $\otimes$ of D-modules, should be given by a kernel.

Obvious examples: (1) M any variety $\rightsquigarrow \delta_{Diag} \subset M^3$ (direct image of Δ)

(induces usual $\otimes$ on D-modules).

(2) $M = G$ group scheme e.g. $\mathbb{A}^1$, $\mathbb{A}^1 \setminus 0$, abelian variety

G, $\mathbb{G}_m$

gp-scheme, then take δ graph (group law), induces $\otimes$ by convolution.

(2)' $G \supset H$ finite gp; & take $M = G/H$ (project group law to quotient).

(also, one can marry 1+2; make a family of group laws depending on parameters ("fiberties w/ group fibers"))

(first non-trivial Hitchin system: SL_2 loc. system w/ unipotent monodromy around)

Nontrivial examples: $\{0, 1, s, \infty\} \subset \mathbb{P}^1$, $s \in \mathbb{C} - \{0,1\}$ given.

$M = \mathbb{A}^1$ (0 of addition; $= E/_{x \mapsto -x}$ involution)

Introduce $P(x,y,z) = (xy + yz + zx - s)^2 + 4xyz(1+s-(x+y+z))$.

The multiplication kernel:

(*) $K := P(x,y,z)^{-1/2} dx\,dy$: Left $D_{\mathbb{A}^1}^{(z)}$-module, right $D_{\mathbb{A}^1}^{(x,y)}$-module

$\{P = 0\}$ ~~is a hypers~~ is a hypersurface $\subset \mathbb{A}^3$.

(consider $E/_{\{\pm 1\}}$, group law, project down, P hypersurface of zeroes of mult. kernel, for addition law of E)

But: this group law isn't just $\delta|_{\{P=0\}}$, but involves this $P^{-1/2}$ factor; some twist.

Geom. Langlands correspondence:

$\otimes$ functors from $(\text{Hol } D_M\text{-mod}, \otimes) \longrightarrow D^b(\text{Vect})$

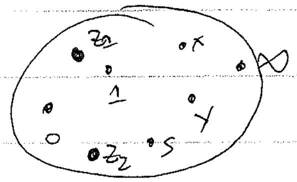
$\longleftrightarrow$ points of N. This should be given by expressions like this: "Hecke eigensheaves"

$$\mathcal{E} \in \text{Hol}_{D_{mod}} \mapsto R\Gamma_c(\mathcal{E} \otimes \mathcal{L}_\lambda)$$

where $\mathcal{L}_\lambda \in \text{Hol } D_M\text{-mod}$, $\lambda \in N$.

The fact that this is a $\otimes$ functor reduces to:

$$\boxed{\forall \lambda, \ (pr_3)_!(\text{mult} \circ pr_3 \mathcal{L}_\lambda) = pr_1 \mathcal{L}_\lambda \otimes pr_2 \mathcal{L}_\lambda}$$



Pick two other points x, y, & fix

$\rightsquigarrow P$ becomes quadratic in z.

zeroes of $P(x,y,-)$. Remarkable coincidence: get two ropes out; should be (of z & of $\mathcal{L}$)

Same should be the (pure topological identities) should hold for ℓ-adic sheaves, mysterious to prove directly.

Introduce

$$H_1 = \frac{\partial}{\partial t} t(t-1)(t-s)\frac{d}{dt} + t \quad \text{in variable } t.$$

Then, equality: $H_1^{(z)} K = K H_1^{(x)} = K \cdot H_1^{(y)}$ (apply H on right or left).

(common eigenvectors of Hamiltonians H, $\rightsquigarrow$ eigensheaves?)

($\rightsquigarrow$ get underdetermined system, should extend to determined system?)

(people tried SL_2, $\mathbb{Q}$. w/ not necessarily unipotent monodromy — in general don't understand how to write the formula!)

<u>Ex:</u> $\mathbb{A}^1$, + convolution: $\rightsquigarrow$ parametrizes "additive characters" of field.

$\mathbb{G}_m$, $\times$ convolution: $\rightsquigarrow$ "multiplicative characters"

Γ function: function of two variables

$$(\mathbb{A}^1, +) \times (\mathbb{G}_m, \times)$$

[Barnes identity]:

$$\int \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\,ds = \frac{\Gamma(a+c)\Gamma(b+c)\Gamma(a+d)\Gamma(b+a)}{\Gamma(a+b+c+d)}$$

appropriate contour

identity btw. transcendental stuff,

but in some sense, compare purely algebro-geometrically!

Version of int. system / finite fields $\mathbb{F}_q$.

M variety $/ \mathbb{F}_q$.

mult. kernel $\in$ $D^b_{constr?}(M, \overline{\mathbb{Q}}_\ell)$ $\in$ motivic constructible sheaf

ℓ-adic sheaf

or motivic sheaf

$\otimes$: Cat : objects are varieties / k.

any field k & $\mathrm{Hom}(V_1, V_2) = K^0$ (motivic sheaves on $M \times N$). linearized version (later, full category).

Composition: pull back, pushforward w/ finite support.

$k = \mathbb{F}_q$: $C_k \xrightarrow{\phi_q} \mathbb{Q}$ - spaces (trace of Frobenius, — — & comp. $\mapsto$ multiplication of vectors).

$M \longrightarrow \mathbb{Q}^{M(\mathbb{F}_q)}$.

M comm. assoc.

$\rightsquigarrow$ $(\mathbb{Q}^{M(\mathbb{F}_q)})$ comm. assoc. / $\mathbb{Q}$.

Usually on M have involution σ_∞ (frobenius?)

& trace: $t: M \longrightarrow \mathbb{1}$ non-deg. w/ $\mathrm{tr}(ab^*)$ pos. def. form on the nose.

$\sum_{x \in M(\mathbb{F}_q)} \mathrm{Tr}_{F(q)}$ ————

Def: M multiplication kernel $(\mathbb{F}_{q^n})$ $n \geq 1$ 'points of M'

$= \mathrm{Hom}_{alg/\mathbb{Q}}(\phi_{q^n}(M, mult), \overline{\mathbb{Q}})$ ("multiplicative characters).

(can apply for $\mathbb{F}_{q^n}$

$\mathbb{F}_q$ points $\hookrightarrow$ $\mathbb{F}_{q^n}$ points should induce a homomorphism of algebras

$\phi_q(M, n) \overset{?}{\longleftarrow} \phi_{q^n}(M, n)$

$b_2 \in M(\mathbb{F}_q)$ For each $b_1 \in M(\mathbb{F}_{q^n})$, compute $n_{b_1 b_2}$.

$\mathrm{mult}_2 \in D^b_{constr.}(M^2_{/S_2} \times M)$, S_2 (involution)-invariant.

For each n, get $\mathrm{mult}_n \in D^b_{constr}(M^n \times M)$, invariant under S_n - use equivariance, Frobenius twists, to produce

Eventually, $\overline{\mathbb{F}}_q$-points. & on this, get action of

$$\mathrm{Frob}_{\overline{\mathbb{F}}_q/\mathbb{F}_q} \times \mathrm{Gal}\left(\overline{\mathbb{Q}}^{CM}/\mathbb{Q}\right).$$

(Declare motivic sheaves on dual space as motivic sheaves on original space, & when calculate eigenvalues, apply dual. —)

(The claim: one can extract for mult. kernels Drinfeld type Langlands from gen. Langlands?)
(Same identity for Γ fns. over $\mathbb{F}_q$ ...)

Generalized Langlands corresp. / $\mathbb{F}_q$.

Say (M, mult. motivic) / $\mathbb{F}_q$.

① Eigensheaves def. / $\mathbb{F}_q$ $\mathcal{L}_\lambda \in$ motivic sheaves $(M) \otimes \overline{\mathbb{Q}}$

s.t. $pr_{12}(\mathrm{mult} \circ pr_3^* \mathcal{L}_\lambda) = \mathcal{L}_\lambda \boxtimes \mathcal{L}_\lambda$.

"finite field analogue of Hecke eigensheaves"

?

② class in K_0 (motivic sheaves on M).

gives homom. of rings numerically.

③ a homomorphism $(\mathbb{Q}^{M(\mathbb{F}_q)}, \text{mult.}) \longrightarrow \overline{\mathbb{Q}}$

Conj: all arrows are identities.

③ → ①: ∃ finitely many sheaves of $\mathbb{F}_q$, take direct sum of all, & use this action) & check it's a Hecke eigensheaf

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Ex. M = abelian variety.

$\chi : M(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}$ characters.

$\mathcal{L}_\chi \rightsquigarrow$ Hecke eigensheaf

(Now, given $x \in M(\mathbb{F}_{q^n})$

$$\mathrm{Tr}_{Fr_x}(\mathcal{L}_\chi) = \chi(x + F(x) + \cdots + F^{n-1}(x))$$

Take $\displaystyle\sum_\chi$ — $= \begin{cases} \# M(\mathbb{F}_q) & \text{if } x + F(x) = 0 \\ 0 & \neq 0. \end{cases}$

$= \{ \# y \in M(\mathbb{F}_{q^n}) \mid x = y - F(y) \neq 0 \}$.

The same formula works in general for an integrable system.

A alg — $F : A \circlearrowleft$ Frobenius act.

$\rightsquigarrow$ a functional $a \longmapsto A \longmapsto \mathrm{Trace}(F \circ (a * -)))$

(coincides exactly w/ direct sum of Hecke eigenspaces).

Q: why does A act on Hecke eigenspaces?

Given $b \in A$ with $F(b) = b$, then b commutes w/ composition as functors,

so get two commuting functors

(another point: using mult. kernel, can explicitly write Drinfeld moduli spaces —)