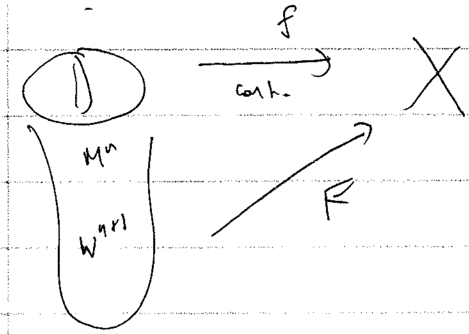
J. Morgan



$\rightsquigarrow \Omega_n(X)$

V^n oriented n-mfld

$[V]^n$ Borel-Moore cycle, &

$$\frown [V] : \quad C^*(V;\mathbb{Z})[n] \xrightarrow[q.\simeq]{} R\,\mathrm{Hom}\,(C_c^*(V;\mathbb{Z}),\mathbb{Z})$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad C^{BM}_{-*}(V;\mathbb{Z}),$$

on homology, gives $H^{*-n}(V;\mathbb{Z}) \to H^{BM}_{-*}(V)$.

When V compact, becomes usual

$$H^{n-*}(V;\mathbb{Z}) \xrightarrow{\cong} H_*(V;\mathbb{Z}) \quad \text{P.D.}$$

Inj. resolution of $\mathbb{Z}$: $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ for computing

$R\,\mathrm{Hom}$ gives usual UCT statement:

$$0 \to \underbrace{\mathrm{Hom}(\mathrm{Tor}\,H^{*+1}(V),\mathbb{Q}/\mathbb{Z})}_{\text{Tor sub gp.}} \to H_*(V,\mathbb{Z}) \to \mathrm{Hom}(H^*(V),\mathbb{Z}) \to 0$$

(switched H^* & H_*, but no big deal).

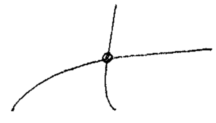
$\Rightarrow$ two different pairings:

(•) $\quad H^{n-*}(V) \otimes H^*(V) \to \mathbb{Z}$ from $\qquad$ <u>intersection</u> pairing

& (••) $\mathrm{Tor}\,H^{n-*}(V) \otimes \mathrm{Tor}\,H^{*+1}(V) \to \mathbb{Q}/\mathbb{Z}$. $\qquad$ <u>linking pairing</u>

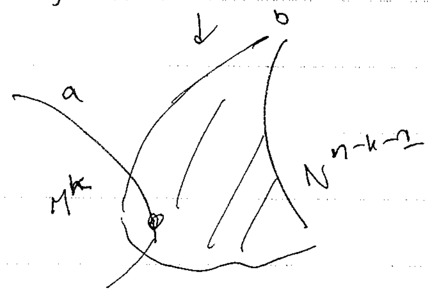
in fact, both <u>perfect pairings</u> (for (•) need to divide by torsion; dual lattices.
for (••), take Pontryagin dual torsion).

(•)



intersect cycles

Linking pairing: Take B, $\partial B = n\cdot a$, & take $\frac{1}{n}(B\cdot a) \in \mathbb{Q}/\mathbb{Z}$ ..



torsion, so some multiple of N bounds

& these pairings are (graded) symmetric:

$$a\cdot b = (-1)^{|k||b|}\, b\cdot a .$$

$$\ell(a,b) = (-1)^{(a+1)(b+1)}\, \ell(b,a) .$$

These two are equivalent to P.D. isomorphisms.

Abstracting out P.D.: Look at chain cplxes with

$$C^*[n] \xrightarrow[q.\simeq]{\phi} R\mathrm{Hom}(C^*, \mathbb{Z}).$$ (w/ symmetry, as in case of mfolds)

"P.D. co-chain cplexes of degree n"

quotient: $(C^*, \phi) \simeq 0$ if there is a "Lefschetz self-dual triple" with (C^*, ϕ) as boundary

What are the P.D.(n) groups?

Lemma:

$$PD(n) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \ (4) \\ \mathbb{Z}/2 & n \equiv 1 \ (4) \\ 0 & n \equiv 2 \ (4) \\ 0 & n \equiv 3 \ (4). \end{cases}$$

The invariants detecting are

(•) signature in $0\ (4)$, and (••) deRham invt. in $1\ (4)$.

What's (•)? Easier. Given a P.D. cplx in $n = 4k$, have induced

$$H^{2k}(C^*) \otimes H^{2k}(C^*) \to \mathbb{Z}$$

unimodular & symmetric (b/c $2k$). Such pairings have a signature.

If there is a Lefschetz triple, note that

$$H^{2k}_{abs}(W^\circ) \longrightarrow H^{2k}(C^*) \longrightarrow H^{2k+1}_{rel}(W^\circ, C^\circ)$$

dual & maps are dual

$\Rightarrow \mathrm{Im}\, H^{2k}_{abs}(W) = $ Lag'n subspace $\Rightarrow$ Signature is zero.

Say A is a f.d. abelian group w/

$lk : A \otimes A \to \mathbb{Q}/\mathbb{Z}$ which is skew, non-degenerate, meaning

$$lk(a,b) = -lk(b,a)$$

Ex: $A = \mathbb{Z}/2\langle x\rangle$, & $lk(x,x) = 1/2$. Note it has no Lag'n subspace, b/c its order isn't square.
but
$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1/2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$2x$ it has a Lag'n subspaces.

So ~~lk~~ deRham invt. is simply "counts # of $\mathbb{Z}/2$ factors mod 2." This is the only issue.

Similar arguments shows that given W, deRham invariant is 0.

It's clear that aside from dim 2, all these P.D. cplxes are realized.

($\exists$ 5-mfld w/ lk 1 by Smale in 60's, & can stabilize!)

Last thing: If have ch. cplx whose invariants are 0, it's a "boundary."

Pf: (for n even).

Given $C^*[n] \xrightarrow{\phi} R\mathrm{Hom}(C^*, \mathbb{Z})$ $n = 2k$

Completely general statement: very easy to get everything but self-paired part..

We can assume $H^* = 0$ for $* \neq k$, and H^k free abelian.

Have

$$H^k \otimes H^k \longrightarrow \mathbb{Z} \quad \text{symm. or skew symm.}$$

If skew symm or symm. w/ signature 0, $\exists$ Lag'n subspace. $L \subset H^k$. So then get a Lefschetz seq:

Take: $H^k_{abs}(W) = L$

$H^{k+1}_{rel}(W, C) = L^*$, & have

$$L \hookrightarrow H^k \xrightarrow{i^*} L^*.$$

This is the exact sequence.

Same argument for $\mathbb{R}$ case.

Dimension $4k+3$ interesting; most don't have Lag'n subspace,

but all realized by 3 mfolds, & all 3 mfolds bound, so can modify any pairing to one + its dual, which has a Lag'n subspace

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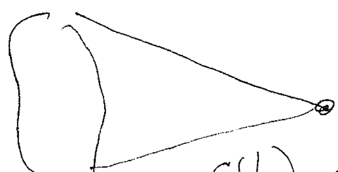
Suppose have



$C(L)$ ← allowed? (*)

L^{2l}

If $L = \partial c(L)$, $\Rightarrow$ $H_k(L)$ has a Lag'n subspace for $\langle -, - \rangle$.

Can allow the cone iff we pick out a distinguished Lag'n subspace of $H_k(L)$.

Rmk: ($\mathbb{CP}^2$ can't bound in this theory, b/c signature is an invariant!)

--- Following Goresky–MacPherson. Pick Lag'n subspace $S \subset C_*(L)$.

$C_*(cL) := C_*(L) + (\text{cones on certain chains on } L)$

Allow $c(\xi)$ if $\dim \xi > k$, or if

$\dim \xi = k$, $[\xi]$ closed, and $[\xi] \in S$.

$$H_*(L) \to H_*(cL)$$

$$H^{abs}_*(cL) = \begin{cases} 0 & * > k \\ H_k(L)/S & * = k \\ H_*(L) & * < k \end{cases}$$

& $H^{rel}_*(cL, L)$ is opposite; satisfies a Lefschetz dual sequence.

Rmk: Can't always make these choices of S; & result depends on S!

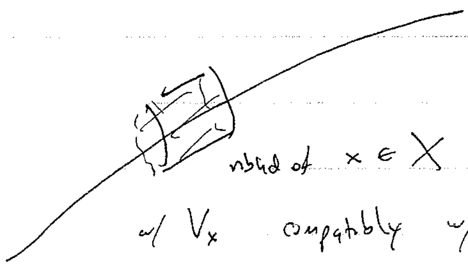
(this is the local transverse picture to a stratum (*) for stratified spaces).

X stratified space. (n-dimensional)

$X = X_n \supset X_{n-1} \overset{\text{no codim. 1 stratum}}{=} X_{n-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$.

where $X_i \setminus X_{i-1}$ smooth mfld dimension i.

$X_n \setminus X_{n-1}$ oriented, &



$X_i \setminus X_{i-1}$ strata dim. i.

"link", compact, self-stratified.

nbhd of $x \in X$ is of the form $V_x = B^i_\epsilon \times c(L^{n-i-1})$

w/ V_x compatibly w/ stratified structure

stratum preserving.

$\mathcal{F}$ bounded complexes of sheaves that are coh. constructible: $\mathcal{H}^*$ are loc. system on each stratum, fin. gen. stalks.

X Hausdorff, loc. cpct, coh. finite dimensional (e.g. stratified, triangulated)

Then, There is a bonded cplx of sheaves, called D_X the Verdier dualizing complex

[X stratified $\Rightarrow$ D_X coh. constructible]

Properties:

1) $\forall\ U^{\text{open}} \subseteq X$, $\quad \Gamma(U, D_X) \underset{q.\text{iso}}{\simeq} \underbrace{R\mathrm{Hom}(R\Gamma_c(U, \underline{\mathbb{Z}}), \mathbb{Z})}$ (cpctly supported section)

on homology, $\mathbb{H}^*(U, D_X) \qquad H^{BM}_{-*}(U, \mathbb{Z})$.

(X a manifold, $D_X = \underline{\mathbb{Z}}[n]$. But in general D_X a complex).

2) For any $\mathcal{F} \in D^b(X)$, define.

$D_X(\mathcal{F}) = \mathcal{H}om(\mathcal{F}, D_X)$. Then, have an isomorphism

$R\mathrm{Hom}(R\Gamma_c(U, \mathcal{F}), \mathbb{Z}) \xrightarrow{\ \sim\ } \Gamma(U, D_X(\mathcal{F}))$.

cohomology: $\quad H^{BM}_{-*}(U, \mathcal{F}) \xrightarrow{\ \sim\ } H^*(U, D_X(\mathcal{F}))$.

e.g. $H_*(X, \mathcal{F}) \xrightarrow{\cong} H^*(X, D_X(\mathcal{F}))$.

Say sheaf ξ satisfies duality dimension n

if $\phi: \xi[n] \xrightarrow[\text{q.iso}]{\cong} D_X(\xi)$.

($\Rightarrow$ usual local P.D.

For cpt X, $H^*(X, \xi)$ satisfies P.D. dim. n).

Let $U_i = X \setminus X_{n-i-1}$. codim $\geq i$.

$U_0 = X \setminus X_2$ smooth oriented mfold. records orientation

Use $\underline{\mathbb{Z}}_{U_0}$. Note $D_{U_0} \cong \underline{\mathbb{Z}}_{U_0}[n]$, &

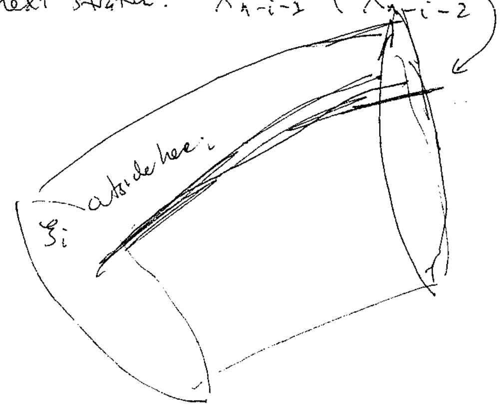
$\phi: \underline{\mathbb{Z}}_{U_0}[n] \xrightarrow{id} D_{U_0}$.

Work inductively over the U_i, increasing i.

Suppose we have ξ_i over U_i &

$\phi_i : \xi_i[n] \xrightarrow{\sim} D_{U_i}(\xi)$ (& a symmetry cond. suppressed; this htpic to its dual)

next stratum: $X_{n-i-1} \setminus X_{n-i-2}$



extend by zero, maximal extension, intermediate extension.

$j_! \xi_i \longrightarrow Rj_* \xi_i \longrightarrow i_*(Rj_* \xi_i)$

loc. const

w/ $H^j = H^j(L_x, \xi_i)$

hypercohomology of link of stratum.

Satisfies duality by inductive hypothesis

Want to kill off half cohomology.

Deligne ξ $\qquad S \subset \mathcal{H}^i(\xi)$

have $\tau_{\leq S}(\xi) \hookrightarrow \xi$ (Amazing: well defined in derived category)

↑ everything below i, & S in degree i.

Use this:

Take

for S local system of Lag'n subspaces of sheaves.

$$
\begin{tikzcd}
 & \xi_{i+1} \arrow[r] \arrow[d] & i_* \tau_{\leq S}(i^* Rj_* \xi) \arrow[d] \\
i_! \xi_i \arrow[r] \arrow[ur] & Rj_* \xi_i \arrow[r] \arrow[dr] & i_*(i^* Rj_* \xi_i) \arrow[d] \\
 & & \text{quotient}
\end{tikzcd}
$$

Can use this to make a bordism theory... & an extraordinary homology theory w/ coefficients exactly as given

(to check Mayer-Vietoris, use "Whitney condition" / Whitney stratified space)

when you make periodic, dual to $K_{\mathbb{R}}$ at odd prime

is $p=2$, surgery relates

(Q: what's the topology of space of choices of Lag'ns ?

pairing on Lag'ns ~)