D. Orlov, Noncommutative varieties and ~~their~~ geometric realizations.

k - base field.

DG algebra $\mathcal{E} = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}^i$ $\quad d : \mathcal{E}^i \to \mathcal{E}^{i+1}$, $d^2 = 0$, $d(ab) = (da)b + (-1)^{\bar{a}} a\, db$

$\mathcal{E} \longmapsto D(\mathcal{E})$ (derived cat of modules) triangulated category.

$= H^0(\mathcal{M}od\text{-}\mathcal{E}) / H^0(Ac\text{-}\mathcal{E})$, where

$\mathcal{M}od\text{-}\mathcal{E}$ = DG category of all DG modules

$\cup$

$Ac\text{-}\mathcal{E}$ - acyclic DG modules

$\exists$ a small triang. subcat.

$Perf(\mathcal{E}) \subset D(\mathcal{E})$. Two definitions:

$Perf\, \mathcal{E} := D(\mathcal{E})^c$ (cpct objects).

$\because$

{the smallest tr. subcat. in $D(\mathcal{E})$ that contains $\mathcal{E}$ & closed under taking direct summands}

There is an enhancement; $D(\mathcal{E}) := H^0(SF\text{-}\mathcal{E})$, where
$SF\text{-}\mathcal{E} \subset \mathcal{M}od\text{-}\mathcal{E}$ "semi-free modules" DG subcat.

and moreover, there's an induced enhancement of ~~Perf~~ $\mathcal{E} = H^0(\mathcal{P}erf\, \mathcal{E})$.

Def: A (derived) non-commutative scheme is a DG category of the form $\mathcal{P}erf\, \mathcal{E}$ where $\mathcal{E}$ is a DG algebra that is coh. bounded.

In this case, $D(\mathcal{E})$ — 'derived category of quasi-coherent sheaves.'

Main idea: when we have $X \longmapsto \mathcal{P}erf\, X$ ~~generalized~~. generalized.

e.g.) Thm: (Keller, Neeman, —). Let X be a quasi-compact and separated scheme.

Then, $D(Qcoh\, X) \cong D(\mathcal{E})$

$\exists$ DGA $\mathcal{E}$ s.t. $Perf\, X = Perf(\mathcal{E})$, w/ $\mathcal{E}$ cohomologically bounded.

1) A is an algebra, then $D(\mathrm{Mod}\text{-}A) = D(\mathcal{E})$, where $\mathcal{E} = A$.
"affine n.c. schemes."

2) projective nc-varieties. (applying Serre's approach to coh (projective))

$$A = \bigoplus_{i \geq 0} A_i \rightsquigarrow \mathrm{QCoh}(\mathrm{Proj}\, A) = \mathrm{Gr\,Mod}\text{-}A / \mathrm{Tors}\, A.$$

torsion modules: any power annihilated by a power of augmentation ideal!

It can be shown that under reasonable (f.g.-type) cond. on A,

$$D(\mathrm{QCoh}(\mathrm{Proj}.A)) = D(\mathcal{E})$$
~DG algebra (find some generators)

(Artin-Tate-VdB): $\mathbb{P}^2_{\mathcal{M}} \rightsquigarrow A = T^\bullet V / I$ $\quad \dim V = 3$

^nc defi. of $\mathbb{P}^2$ $\quad I \subset V \otimes V$ quad. relations,

where $\dim I = 3$.

Then, $D^b(\mathrm{coh}\, \mathbb{P}^2_{\mathcal{M}}) = \mathrm{Perf}\, \mathbb{P}^2_{\mathcal{M}} = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$

$$\underbrace{\bullet \overset{u}{\Rrightarrow} \bullet \overset{v}{\Rrightarrow} \bullet}_{W}$$

w/ $\dim U = \dim V = 3$
& $\dim W = 6$,

& $\mu : U \otimes V \to W$.

& so here $W = U \otimes V / I$. (the moduli of these $\mathbb{P}^2_{\mathcal{M}}$ is the moduli of these)

We have $\mathrm{Perf}\, \mathcal{E}$, $(\mathcal{E} = \bigoplus_{i \in \mathbb{Z}}, d)$. How to see it's proper?

Def: $\mathrm{Perf}\, \mathcal{E}$ is <u>proper</u> if $\dim_k \bigoplus_i H^i(\mathcal{E}) < \infty$.

$\Longleftrightarrow$ for any two $U, V \in \mathrm{Perf}\, \mathcal{E}$, $\dim_k \bigoplus_i H^i(\mathrm{Hom}^\bullet(U,V)) < \infty$.

[Kontsevich]
Def: $\mathrm{Perf}\, \mathcal{E}$ is smooth if $\mathcal{E}$ as a bimodule is perfect.

$\Longleftrightarrow$ $\mathcal{E}$ is perfect as $\mathcal{E} \otimes \mathcal{E}^{op}$ module.

Rmk: In usual geometry, smoothness depends on k base field.

Def: Perf $\mathcal{E}$ is regular if $\mathcal{E}$ is a strong generator for Perf $\mathcal{E}$.
(roughly about "generation time" by $\mathcal{E}$ being finite, for any object in Perf $\mathcal{E}$ (~~bounded~~ uniformly bounded?)).

(Rmk: smooth $\Rightarrow$ regular?)

Let X be a smooth projective variety, & consider $j: \mathcal{N} \hookrightarrow$ Perf X full subcat.
$\mathcal{N}$ is called admissible if j has right and left adjoint functors.
(meaning has a right & left projection).

Say $\mathcal{N} \subset$ Perf X admissible. Immediate that $\mathcal{N}$ is proper, but it's also regular, whenever Perf X is. (by applying these adjoints to ~~generation~~ triangles —)

$\rightsquigarrow$ $\mathcal{N}$ is a smooth proper noncommutative scheme.

Def: E is exceptional if $\mathrm{Hom}(E,E) = k$ & $\mathrm{Hom}(E, E[m]) = 0$ $m \neq 0$.

$\rightsquigarrow$ $\underset{\underset{D(\mathrm{pt.})}{\cong}}{\langle E \rangle} \subset$ Perf X

Def: Let Perf $\mathcal{E}$ be a nc scheme. ~~The~~ A geometric realization of it is a realization as a full subcategory in Perf X where X is smooth and projective.

$\rightsquigarrow$ Perf $\mathcal{E} \xrightarrow{\sim} \mathcal{N} \subset$ Perf X.

(admissible is a property of Perf $\mathcal{E}$ to be not just proper but smooth?).

$\exists$ interesting examples:

1) $Y \subset \mathbb{P}^{n+1}$ n-dim'l smooth cubic.

$\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-2) \rangle$ is exceptional

$\Rightarrow \exists\, \mathcal{N}$ w/

$\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-2), \mathcal{N} \rangle = \mathrm{Perf}(Y)$

Thm: If $n = 3x+1$, then $\mathcal{N}$ is a nc. CY variety.

(Rmk: If have $\mathcal{N} \subset$ Perf X, admissible have ${}^{\perp}\mathcal{N} \subset$ Perf X, w/ a s.o. decomp
Perf $X = \langle \cancel{\mathcal{N}}\, {}^{\perp}\mathcal{N}, \mathcal{N} \rangle$ & ${}^{\perp}\mathcal{N}$ also admissible!)

e.g. when $n=4$,

$\mathcal{N}$ is a nc K3 surface; 20-dim'l family of nc K3's.

(Lattice $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$).

$n=7$: $\mathcal{N}$ is a 3d n.c. CY variety which is not a deformation of commutative CY variety.

$\mathcal{N}$ is D-branes of type B on a LG model on $(\mathbb{C}^{n+1}, W)$?

where $W^{-1}(0) = Y$.

2) Gushel-Mukai variety : $\underbrace{G(2,5) \cap \overset{\text{quadric}}{Q} \cap \overset{\text{hyperplane}}{H}}_{Y \ \text{4-dim'l}}$.

Have exceptional collection:

$$\langle \mathcal{O}, \mathcal{E}, \mathcal{O}(1), \mathcal{E}(1), \mathcal{N} \rangle$$

2d v.b. which is anticanonical on $G(2,5)$

NC. K3 surfaces; 20-dim'l family (lattice: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$).

Conjecture: There are infinitely many (countably many) 20-dim'l families of nc-K3 surfaces (algebraic varieties). More precisely, they are related to positively defined 2-dim'l even lattices $\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$ w/ $4ab - c^2 > 0$.

3) phantoms (quasi-phantoms): by def'n,

$\mathcal{N} \subset \mathrm{Perf}\, X$ admissible s.t. $\left(\begin{matrix} HH_*(\mathcal{N}) = 0 \\ \& \ K_0 = \text{trivial} \end{matrix} \right)$, and for quasi-phantom, $K_0(\mathcal{N})$ finite group.

They exist, and moreover, for phantoms,

$K_*(\mathcal{N}) = 0$.

How to construct them?

"fake del Pezzo surfaces" & big exceptional collection

$\mathrm{Perf}\, S = \langle \mathcal{L}_1, \ldots, \mathcal{L}_N, \mathcal{N} \rangle$ ← phantom.

(So many ~~big~~ examples come w/ geom. realizations. In a reverse direction, we'll ask abt. the existence of geom. realizations. (& $\exists$ one $\Rightarrow$ $\exists$ many geom. realizations)).

Q: Is there a geom. real. for $\mathrm{Perf}\, \mathcal{E}$ ($\mathcal{E}$ smooth proper nc variety).

Unfortunately at this level no general answer.

Suppose $\mathrm{Perf}\, \mathcal{E} = \langle E_1, \ldots, E_n \rangle$ has a full exceptional collection.

Conj: If $\mathrm{Perf}\, X$ has a full exceptional collection, then X is rational.

Conj: There are no phantoms in such ~~categ~~ $\mathrm{Perf}\, \mathcal{E}$.

(this would imply that any excep. coll. of right length would be full too) hard/under, but important b/c

Thm: Let $\mathrm{Perf}\, \mathcal{E} = \langle E_1, \ldots, E_n \rangle$ full excep. coll'n. Then,

$\mathrm{Perf}\, \mathcal{E} \cong \mathcal{N} \subset \mathrm{Perf}\, X$, X sm. proj. variety.

(the proof is constructible : X can be realized as a tower of projective bdles!

Thm: A a fin. dim'l algebra s.t. $S \neq A/_{a}R$ "semisimple part" is separable.

(trivial for perfect fields $\mathbb{k}$). Then there is a sm. proj. X w/ $F: \mathrm{Perf}\, A \hookrightarrow \mathrm{Perf}\, X$.

If A has fin. glob. dimension, then $\mathrm{Perf}\, A$ is admissible.

The existence of $F \longleftrightarrow \exists\, u \in \mathrm{Perf}\, X$ s.t. $\mathrm{End}^{\circ}(u) = A$ &

& $\mathrm{Hom}(u, u[m]) = 0$ $m \neq 0$.

Under these situations is it possible to find X and $\mathcal{U}$ such that $\mathcal{U}$ is a vector bundle?

When $A = kQ/I$, Q-quiver $Q = (1, \ldots, n)$ ordered meaning $\forall$ arrows ~~A~~, $S(A) \le T(A)$ (source target)

In this case, Perf $A = \langle P_1, \ldots, P_n \rangle$ (vertices). In this situation, we also have:

<u>Thm</u>: Let A be a quiver algebra on n ordered vertices. Then $\exists$ X sm. proj. &

$$u: \underset{\text{f.d. modules}}{\text{mod-}A} \longrightarrow \underset{\text{abelian cat.}}{\text{coh } X} \quad \text{s.t.}$$

1) $u: D^b(\text{mod-}A) \to D^b(\text{coh-}X)$ fully faithful.

2) simples $S_i \longmapsto \mathcal{L}_i$ line bdles on X.

3) Any A-module M goes to a vec. bdle on X w/ $\dim(M) = \text{rk } u(M)$.

4) X is a tower of proj. bdles

$\Rightarrow$ moduli of reps of a quiver $\hookrightarrow$ moduli of v.b.'s on an X.

These have interesting consequences. For <u>ex</u>:

$\mathbb{P}^2_u$

$\mathbb{P}(\mathcal{F}_u)$ comm. deformation of $\nearrow$ $\text{Hilb}^2 \mathbb{P}^2 = \mathbb{P}_{\check{\mathbb{P}}^2}(S^2 \mathcal{T})$

two parts connected by c.le.

$\downarrow$ dual $\mathbb{P}^2$

$\check{\mathbb{P}}^2$

$\mathcal{F}_u$ nc. deformation

$\parallel$

$\langle D^b(\mathbb{P}^2), D^b(\mathbb{P}^2_u), D^b(\mathbb{P}^2_u) \rangle$

so $\mathcal{G}$ can be realized in a comm. variety.

<u>Conj</u>: Any nc deformation of sm. proj. X, X_u, can be realized as a semi-orthogonal part of a <u>commutative</u> deformation.

e.g. can find Y & $\text{Perf } X \hookrightarrow \text{Perf } Y$

s.t. $\underset{\text{nc. def}}{\text{Perf } X_u} \hookrightarrow \underset{\text{c. def.}}{\text{Perf } Y_u}$

Rmk: non-algebraic categories are not cptly gen., so hazardous to consider that setting.