P. Seidel, Differentiating with respect to the Kähler parameter

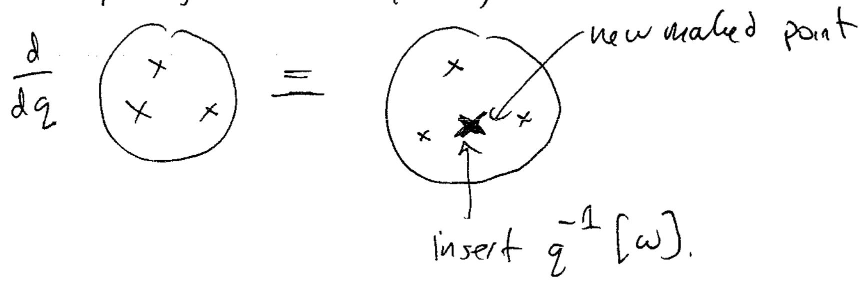
e.g. Gromov-Witten theory

Curves in $A \in H_2$ are counted with $q^{\int_A \omega}$.

Differentiate:

$$\frac{d}{dq}\left(q^{\int_A \omega}\right) = \left(\int_A q^{-1}\omega\right) q^{\int_A \omega}.$$

Schematically using the divisor equation,



Question: what if $q^{-1}[\omega]$ is itself a Gromov-Witten invariant?

The diff'l equations. $\psi \neq 0$.

(1) $$\partial_q \begin{pmatrix} \rho \\ \sigma \end{pmatrix} - \begin{pmatrix} 0 & \psi \\ 4\psi z^{(2)} & \eta \end{pmatrix}\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = 0.$$

(2) Reduction to second order eq'n in ρ (ok b/c $\psi \neq 0$):

$$\partial_q^2 \rho + \partial_q \rho \left(\eta - \frac{\partial_q \psi}{\psi}\right) - 4\psi^2 z^{(2)} \rho = 0$$

(recover σ from ∂_q of ρ & ψ).

(3) $$\partial_q \lambda - \psi\lambda^2 + \eta\lambda + 4\psi z^{(2)} = 0 \quad \lambda = \frac{\sigma}{\rho}$$

(once have fund. sol'n, sth. about family over A^1 or $\mathbb{P}^1$ w/ poles?).

Example: rational elliptic surface.

$$\bar\pi : \bar E \longrightarrow \mathbb{CP}^1.$$ (e.g. $Bl_9 \mathbb{CP}^2$, & simple 1st singularities, but irrelevant.)

Assume $M = \bar\pi^{-1}(\infty)$ smooth ell. curve, so $c_1(\bar E) = [M]$, and

$[\omega_{\bar E}] = [9 \text{ exc. sections}] + \text{constant}[M]$ ($[\omega]_B$ comes from pencil of cubics, —)

fixed large

Also consider $E = \bar E \setminus M$, which comes with $\pi : E \longrightarrow \mathbb{C}$.

Enumerative geometry (extremely well known) $(g=0, n=1)$

section count $z^{(1)} = \sum_{\substack{A \in H_2(\bar E) \\ A\cdot M = 1.}} z_A \, q^{\int_A \omega_{\bar E}} \in H^2(\bar E, \mathbb{K})$

$z_A \in H^2(\bar E)$ the cycle the section spans

$\mathbb{K}$ some field w/ q (e.g. $\mathbb{C}((q))$ or frac. powers, depending on if ω integral).

bisections: $z^{(2)} = \sum_{A\cdot M = 2} \text{---} \in H^0(\bar E; \mathbb{K}) = \mathbb{K}.$

appears in the diff'l eq'ns.

(4) Lemma: $q^{-1}[\omega_{\bar E}] = \psi z^{(1)} - \eta [M]$ for some functions $\psi \in \mathbb{K}^*$, $\eta \in \mathbb{K}$

(almost straightforward; all 3 belong to invt. parts for rich monodromy action, but rk invt. part is 2. $\psi \neq 0$ b/c of lowest order analysis.)

Interesting to say what ψ and η are, but not relevant for talk.

So now diff'l eq'ns well posed, can solve for ρ, σ.

Q: what do ρ, σ mean?

(5) Theorem: Consider the small ~~qu~~ quantum product with $[M] = c_1(\bar{E})$

$$[M] * \_ : H^*(\bar{E}; \mathbb{K}) \to H^*(\bar{E}; \mathbb{K}).$$

12 dim'l.

Then, its eigenvalues are solutions of (3).

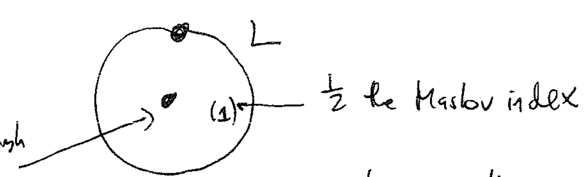
(w/ multiplicity, 12 eigenvalues, but one has mult. 9, so 4 different ones)

(not hard to prove).

(6) "Theorem": Let $L \subset E$ ($\bar{E} \setminus M$) be a cpct Lag'n submanifold w/ 0 Maslov class. The count of Maslov index 2 discs in $(\bar{E}, L)$ is a solution of (3).

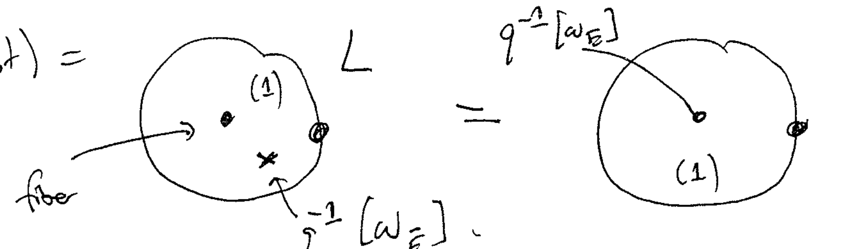
(If $[L] \in H_*(\bar{E})$ non-zero, then $[L]$ is an eigenvector, ~~etc~~ & ... function works. But, this also applies if L nullhomologous!).

"Proof": Count of Maslov index 2 discs:

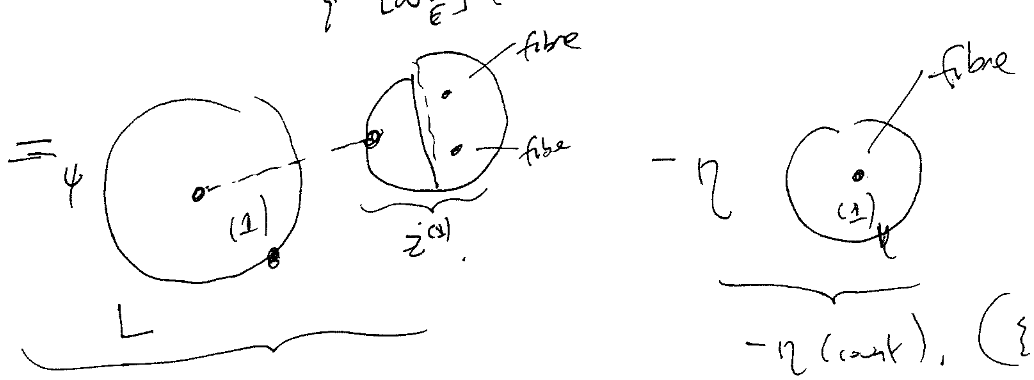


stabilizing fibre (gives stable domain; changes nothing, discs go through fibres once!

$\partial_q(\text{count}) =$

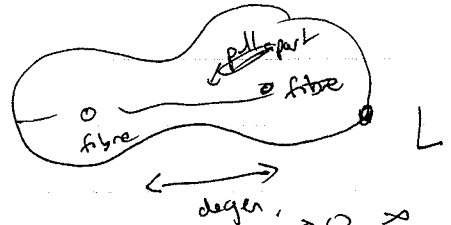


(b/c don't need to stabilize)

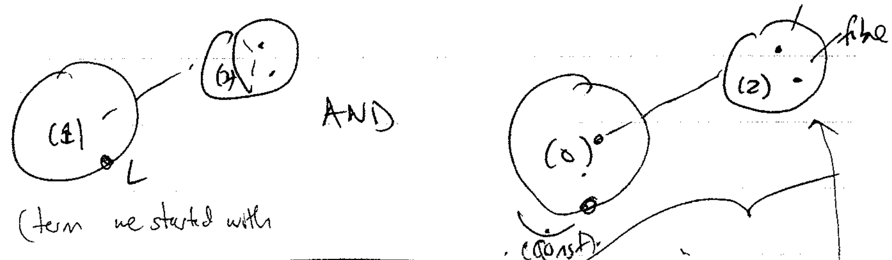


glue

$-\eta(\text{count})$, ($\xi - \eta\lambda$ from (3)),

Ψ 

$= \Psi(\text{const})^2$. Problem: as $r \to 0$, have two terms:



$= -\ddot{\psi}\,\psi\, z^{(2)}$

= const. argument works assuming no Maslov 0 discs (or cancel it out algebraically in the Fukaya category)

divisor eq'n gives $2\times$ every the same fibre for (2).

so get all the terms of (3):

$$\psi\lambda^2,\quad \eta\lambda,\quad \psi\,\psi\, z^{(2)},$$

After minor modification ($[\omega_E]$ = exc. locus & forget adding $[M]$ (not Kähler but still)).
(mult. all by powers of q).

we have

$$\psi = 1 + O(q),\quad \eta = O(q),$$ and (no neg. powers!) we can consider the fundamental solution

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}$$ of (1), $\Theta(0) = \mathbb{1}$.

(7) Thm: Let $\mathcal{B} \subset \mathrm{Fuk}(M)$ be the subcat. of vanishing cycles, then $\mathcal{B}$ is defined over

$$\mathbb{C}[t] \subset \mathbb{K}, \qquad t = \frac{\Theta_{12}}{\Theta_{11}}$$

Novikov

this is the "mirror map" for particular family of ellip. curves where mirror is a "rat'l pencil", therefore polynomial.

polynomial subfield

(in fact, there's a much more precise version of (7) saying exactly how $\mathcal{B}$ deforms).

General picture: Anticanonical Lefschetz pencils,

Start w/ such a pencil (on smooth Fano), & blow-up base locus to get

$$\bar{\pi} : \bar{E} \longrightarrow \mathbb{CP}^1 \quad \text{"graph of pencil"}$$

(family of CY's over $\mathbb{CP}^1$).

~~Lemma~~ (4): $q^{-1}[\omega_{\bar{E}}] = \psi z^{(1)} - \eta[H]$. not true in general.

But: Assume this is true for now (works when Fano has $b_2 = 1$, or some symmetry),

(in general will need B-field for this to hold always, but we'll suppress).

Thm (5) is always true then.

"Thm (6)" "✓" (but have not carried out all technical work)

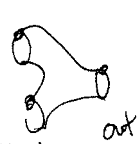
~~Thm~~ (7) is a "Conjecture "polynomiality conjecture for Fuk(CY hypersurface)."

Rmk: in some cases, this is a conclusion of HMS, but should be true in general.

Where do these equations come from? (General framework)

Topological quantum field theory

"parametrized" → param. S^1  ⇝ graded vec. space H^*.

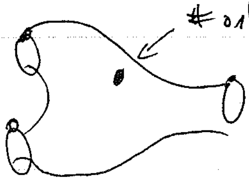
g=0 surfaces  many in, out ⇝ $(H^*)^{\otimes \text{inputs}} \longrightarrow H^*$ (one output).

Similarly for families of surfaces over a closed oriented base P:

$$\rightsquigarrow (H^*)^{\otimes \text{inputs}} \longrightarrow H^* [-\dim P].$$

Finally, allow one additional interior marked point (don't have to have it though)

& could maybe have > 1, but unnecessary for now.



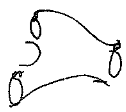
$$\rightsquigarrow (H^*)^{\otimes \text{inputs}} \longrightarrow H^* [2 - \dim P]$$

"codim of insertion"

(formally divisor insertion)

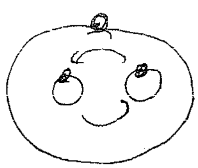
Outcome: is a version of Getzler's analysis.

e.g., have

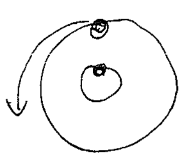


$: * : H \otimes H \longrightarrow H$

$\rightsquigarrow e \in H$



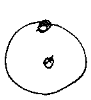
$[-,-] : H \otimes H \longrightarrow H[-1]$



$\rightsquigarrow \Delta : H \longrightarrow H[-1].$ BV algebra

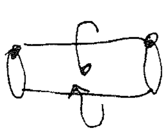
This gives H^* the structure of a BV algebra. In fact, $[x, y] = \Delta(x * y) - (\Delta x) * y - (-1)^{|x|} x * \Delta y$.

extra marked point:



$\rightsquigarrow s \in H^2$, w/ $\boxed{\Delta s = 0}$ by gluing argument.

Similarly:



gives $r : H \longrightarrow H[+1]$

$r = [s, \cdot]$.

So far, not exciting, but

Two refinements: (a) Consider $H = H^*(C, d)$ with chain level operations (a TFT or TCFT).

~~(operations on~~

Have same basic operations, but additional homotopies.

e.g.)

$$s \in C^2, \quad ds = 0, \text{ but now}$$
$$\Delta s = d\sigma \quad \text{for some distinguished } \sigma \in C^0$$

(b) Now assume (C^*, d) is over $\mathbb{K}$ and ~~θ~~ carries a connection

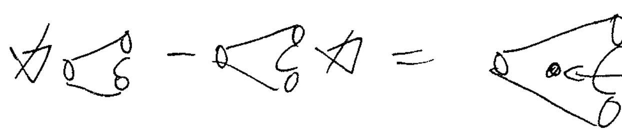
$$\nabla : C^* \longrightarrow C^* \quad \text{in } q \text{ direction:}$$

$$\nabla(f x) = (\partial_q f) x + f \nabla x.$$

This satisfies:

$$\boxed{\nabla d x - d \nabla x = r(x)} \qquad (*) \text{ new axiom}$$

(Think of d as assoc. to -1 dim'l moduli space $=$ 0——0 $/\mathbb{R}$, & r inserts $\circ$. The $/\mathbb{R}$ means only S^1 freedom is left.)

Similarly,  insert anywhere.

(now r is only ch. htpic to $[s, -]$, but can modify so it's actually $[s, -]$.

Think of $[s, -]$ as Kodaira–Spencer class, measures failure/inability to move in q direction.)

(8) Assumption: $[s] \in H^2$ is zero.

Concretely, $s = d\alpha$ for some choice of α. (Singles out a subclass of theories; trivialization of Kodaira Spencer should allow us to move in q-directions.)

Note we then get
$$A = [\underbrace{\Delta\alpha \pm \sigma}_{\text{class b/c } \; d\Delta\alpha = \Delta d\alpha = \Delta s = d\sigma}] \in H^0 \quad \text{(depends on choice of } \alpha\text{!)}.$$

<u>Thm</u>: If (8) holds, H^* can be equipped w/ a connection ∇, which is compatible with $*$ and $[-,-]$ and satisfies
$$\nabla\Delta x - \Delta(\nabla x) = [A, x]. \qquad x \in H^*$$
(not compatible w/ BV structure).

<u>Note</u>: We have a family of connections
$$\nabla^c x = \nabla x - cA * x \qquad c \text{ a scalar.}$$

Now, $\Rightarrow \nabla^{-1}$ is compatible with the BV operator (but not with the product or bracket).

→ Categorical framework: ∇^{-1} natural w.r.t. $HH_\bullet(-)\overset{\mathrm{BV}}{\circlearrowleft}$, ~~HH~~ ∇ natural w.r.t. $HH^\bullet\overset{\mathrm{Gerst.}}{\circlearrowleft}$
if add bending data, ↗.

<u>Geometry</u>:
$$\begin{tikzcd}
\bar{E} \arrow[r, "\bar{\pi}"] \arrow[d] & \mathbb{C}P^1 \\
E \arrow[r, "\pi"] & \mathbb{C} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$q^{-1}[\omega_E] \longmapsto s \in H^*$$
$$\cap \qquad\qquad\qquad \parallel$$
$$H^*(E;\mathbb{K}) \xrightarrow{\quad} SH^*(E)$$

acceleration.

<u>Lemma:</u> $z^{(1)}\big|_E$ (in $\bar{E}$) lies in the kernel of the acceleration map.

Hence, assumption (4) (that $q^{-1}[\omega_{\bar{E}}] = \psi z^{(1)} - \eta[M]$) implies the vanishing of s.

$\Rightarrow$ can get a connection on SH^* in q-direction. (!)

<u>Conjectures</u>

identity $e \in SH^0(E)$ satisfies the diff'l eq'n (2) with ∂_q replaced by ∇^1 ($c = +1$!)

" $\nabla^{-1} \longleftrightarrow$ nat. (id, Serre)

$\nabla^{+1} \longleftrightarrow$ nat (Serre, id), <u>this one has to occur here</u> ".

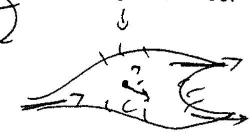
Point: once you have this, and an ∂e formalism, should

$\Rightarrow$ polynomiality conjecture.

<u>Another point:</u> Normally connections live on $HP^\cdot(SH)$, & obstruction to lifting to $SH^\cdot$ is the Kodaira Spencer class. When it vanishes, can lift to $SH^\cdot$ ("trivial family of CY's w/ q-dep. hol. volume form)

which direction to point?

(Rule: ∇^{-1} compat. w/ BV on HH_*^σ

∇^0 compat. w/ HH_σ^0, & ∇^1 compat. w/ CY:  ?)).