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Character variety

$\mathcal{M}$ = Moduli of Rep(π_1(X)) e.g. X = Riemann surface.

$\mathcal{M} = \mathcal{M}_{\text{Betti}}$)

Dual ∂ cplx of M.

D∂M : for a good compactification $M \subseteq \bar{M}$.

Let $D = \bigcup D_i$ be the compl. divisor

simplicial cplx : vertices ⟶ D_i

simplices ⟶ $D_I = \bigcap_{i \in I} D_i$

Thm: htpy type of D∂M is an invt.

Q: what is D∂M for M_B?

Answer for $X = \mathbb{P}^1 \setminus \{y_1, \ldots, y_k\}$

$\mathcal{M} = \mathcal{M}(X, SL_2; C_1, \ldots, C_k)$ $C_i \subseteq SL_2$ conj classes.

$= \text{Hom}^{C_1, \ldots, C_k}(\pi_1(X), SL_2)/\text{conjugacy}$. (boundary monodromy around y_i)

C_i = conj. classes of $\begin{pmatrix} c_i & 0 \\ 0 & c_i^{-1} \end{pmatrix}$ w/ c_i general

The $\mathcal{M}$ = smooth

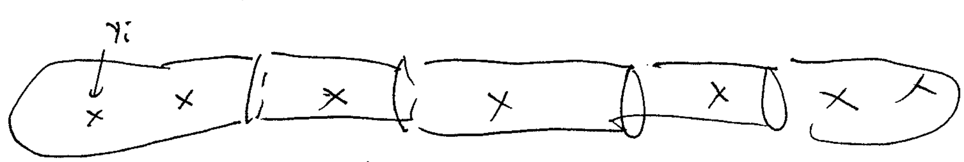
These character varieties are hard to understand.

(Gross–Hacking–Keel–Kontsevich ⟿ general theory?)

Our approach: define algebraic Fenchel–Nielsen coordinates

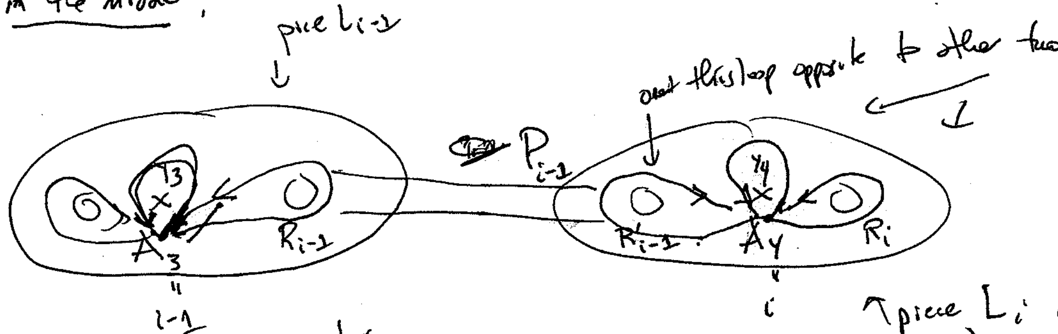
Draw:

X =



each: "three holed sphere" (so ends have two "x", middle have one.)

Gluing in the middle:



↑ piece L_i on S_i.

equations:
- $A_i \in C_i$ ← conj. class. Choose a basis so that $A_i = \begin{pmatrix} c_i & 0 \\ 0 & c_i^{-1} \end{pmatrix}$ (still have 1-dim. "scaling freedom")

$$(*)\quad \begin{cases} \bullet\ R'_{i-1} = A_i R_i, \\ \bullet\ (\text{gluing}):\ R'_{i-1} P_{i-1} = P_{i-1} R_{i-1} \end{cases}$$

(P_{i-1} gluing map intertwines for varieties).

We are going to look at an open subset

$\mathcal{M}' \subset \mathcal{M}$ consisting of rep'ns

s.t. ① $t_i = \mathrm{Tr}(R_i) \neq \pm 2$ (e.g. R_i not $\pm$ unipotent) (know $\det(R_i) = 1$) (maybe removable)

② (more essential).

The loc.-sys. L_i on S_i should be stable w.r. to a choice of parabolic weight at y_i.

$\Longleftrightarrow$ there is no invariant subsystem $F \subset L_i$ w/ c_i^{-1}-eigenvalue

(using the particular parabolic weight singling out c_i^{-1})

(point is: can have an extension, but sub-guy has to be c_i, quotient c_i^{-1}).

$\Longleftrightarrow R_i = \begin{pmatrix} * & \neq 0 \\ * & * \end{pmatrix}$. we fix the basis so

$$R_i = \begin{pmatrix} u_i & 1 \\ * & t_i - u_i \end{pmatrix}.$$

(problem w/ coordinates is that eq'ns depend on all coordinates.

ie., $u_i = \left(\dfrac{t_{i-1} - c_i^{-1} t_i}{c_i - c_i^{-1}} \right)$.
)

Set

$$U_i = \begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}$$

(Can define a canonical matrix w/ trace t_i):

$$T_i = \begin{pmatrix} 0 & 1 \\ -1 & t_i \end{pmatrix}, \quad \& \; R_i = U_i^{-1} T_i U_i .$$

Calculate eq'ns (*):

$$R'_{i-1} = A_i R_i = A_i U_i^{-1} T_i U_i$$

$$= U_i^{-1} (U_i A_i U_i^{-1} T_i) U_i$$

trace is $Tr(R'_{i-1}) = Tr(R_{i-1}) = t_i$.

(1) $$U_i A_i U_i^{-1} T_i = \begin{pmatrix} 0 & c_i \\ -c_i^{-1} & t_{i-1} \end{pmatrix}$$

(To proceed, need to choose a $\sqrt{A_i}$):

Define $$A_i^{1/2} := \begin{pmatrix} c_i^{1/2} & 0 \\ 0 & c_i^{-1/2} \end{pmatrix}$$

Then (1) $= A_i^{1/2} T_{i-1} A_i^{-1/2}$

Conclusion: $$R'_{i-1} = U_i^{-1} A_i^{1/2} T_{i-1} A_i^{-1/2} U_i .$$

Plug into gluing equation: (recalling $R_{i-1} = U_{i-1}^{-1} T_{i-1} U_{i-1}$)

Gluing equation becomes:

$$U_i^{-1} A_i^{1/2} T_{i-1} A_i^{-1/2} U_i P_{i-1} = P_{i-1} U_{i-1}^{-1} T_{i-1} U_{i-1} .$$

multiply by $A_i^{-1/2} U_i$ on left and U_{i-1}^{-1} on right:

$$T_{i-1} \left(A_i^{-1/2} U_i P_{i-1} U_{i-1}^{-1} \right) = \left(A_i^{-1/2} U_i P_{i-1} U_{i-1}^{-1} \right) T_{i-1}$$

So equation becomes:

$$T_{i-1} Q_{i-1} = Q_{i-1} T_{i-1} \quad ; \quad \underline{\text{decoupled}}.$$

(the point is: we should have used Q_{i-1} instead of P_{i-1}. ↑ usual transport map)

Thus,

$$\mathcal{M}' = \{(t, Q)\}^{k-3}_{t \neq \pm 2}.$$

↑ "trace coord." ↑ "glue param" — FN coords:

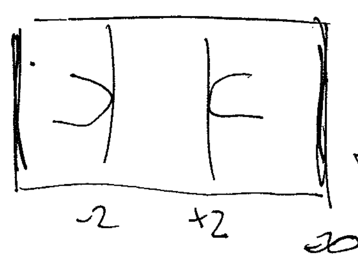
w/ $TQ = QT$, $Q \in PGL_2$, $Q = \begin{pmatrix} p & q \\ p' & q' \end{pmatrix}$

$p' = -q$, $q' = p + qt$, so $Q = \begin{pmatrix} p & q \\ -q & p - qt \end{pmatrix}$ w/ $[p:q] \in \mathbb{P}^1$

$\det Q \neq 0$: $p^2 + tpq + q^2 \neq 0$.

Thus, $\mathcal{M}' = \{(t,Q)\}^{k-3} \,\|\, \{(t,Q)\} = \{ t \in \mathbb{A}^1 - \{\pm 2\}, (p:q) \in \mathbb{P}^1, p^2 + tpq + q^2 \neq 0 \}$

$\{(t,Q)\}$:



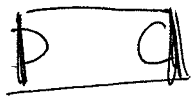
is $\mathbb{P}^1 \times \mathbb{P}^1$ complement of these divisors

$\mathbb{D}\partial($  $) \sim \mathbb{D}\partial($  $)$ (principle: thm of sth. which is $\mathbb{A}^1$-contractible doesn't change $\mathbb{D}\partial$ e.g. these lines only intersect D's in a pt.)

Prop: for same reasons as

$$\mathbb{D}\partial \mathcal{M}' = \mathbb{D}\partial \mathcal{M}.$$

But

$\mathbb{D}\partial$  $= S^1$

(two components meets in two different points
(need to blow up once to get SNC, two components two different points).

Another prop of general theory:

$$\mathbb{D}\partial \left(\text{[see image 2]} \right)^{k-3} = S^{2(k-3)-1} = \mathbb{D}\partial M_{Betti}$$

(rough idea: if $\mathbb{D}\partial = S^1$, then $X, \partial X \simeq \mathbb{D}^2, S^1$,
& $(\mathbb{D}^2)^{k-3}$ is a ball, so its ∂ is a sphere)

General conjecture: (example from 4 yrs ago was $\mathbb{P}^1$ 4 pts.)

$\mathcal{M}_H$ = {Higgs bundles} ⊃ {"link at ∞"}$_H$

↑ "Hitchin" or "Higgs" moduli space

Hitchin map: $\mathcal{M}_H \to \mathbb{A}^N$

{"link at ∞"}$_H$ ↓

$\mathbb{A}^N \leftarrow$ "link at ∞" $= S^{2N-1}$

homeo

Also have

$M_B \leftarrow$ (link at ∞)$_B$

↓

$\mathbb{D}\partial M_B$ (general fact: link at ∞ collapses onto $\mathbb{D}\partial M_B$.)

know (link)$_H$ ≅ (link)$_B$.

conj: these two maps are the same; namely
(a) $\mathbb{D}\partial M_B = S^{2N-1}$,
& (b) the identity map intertwines these vertical maps.

We showed this ~~$\mathbb{D}\partial\mathcal{M}_B$~~ $\ni$

$$\simeq S^{2N-1} \longrightarrow \mathbb{D}\partial\mathcal{M}_B.$$

for SL_2 on $\mathbb{P}^1 \setminus \{y_1, \ldots, y_k\}$, don't show diagram commutes

This is a geometric version of the "P = W" conjecture of Donagi, ...

"perverse Leray filtration on $\mathcal{M}_H$" ↗ P; W ↑ weight filtration on $\mathcal{M}_B$

(known: lowest piece of weight filtration cases for $H^*(\mathbb{D}\partial\mathcal{M}_B)$)

(P=W should say that pull backs of generating cohomology classes are same coh. class)

This is a topological enhancement..

Rmk: $\mathcal{M}_H$ & $\mathcal{M}_B$ definitely don't have same motive; in particular the MHS on them are completely different; some duality btw. them..
?MHS?

e.g. $\mathcal{M}_H$ contains the motives of $Sym^k(\Sigma)$; $\mathcal{M}_B$ indep. of Σ.
Motivic direction of one should corresp. to sth. of other.

[Kapranov]: Are there good natural compactifications? Maybe not --

Higher genus might be possible.

(Rk. 2 local sys. entirely defined by conj. classes of monodromy.
That was the key thing that didn't work for $n>2$ ...)