A. Blanc, Formal deformations of categories (w/ L. Katzarkov, P. Pandit)

Introduction: k field, B associative algebra$/k$,

$k[[t]]$ deformations of B up to iso. $\longleftrightarrow$ $HH^*(B)[1]$ (rather, the complex)

dg Lie algebra. $(\in \mathrm{dgLie}_k)$

$\mathcal{C}$ k-linear ∞-category (presentable, compactly generated)

(ex. A alg., then ...)

(some people include presentable $\Rightarrow$ cptly generated) $\mathrm{Ind}(\mathcal{C}^\omega) \simeq \mathcal{C}$. ($\omega$ = cpct. objects.)

(presentable := accessible localization of $\mathrm{Presheaves}(\mathcal{C}^{small})$)

functors are continuous functors (commute w/ small colims)

e.g., $\mathcal{C} \in \mathcal{C}at^c_k := \mathrm{Mod}_{\mathrm{Mod}_k}(Pr^L)^c$ ← cptly generated.

↑ presentable categories w/ left adjoints

Equivalences are Morita equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$: $\mathcal{C}^\omega \xrightarrow{\sim}_{Morita} \mathcal{C}'^\omega$

Ex: X (derived) scheme $/k$ quasi-compact, quasi-separated

$\mathcal{C} = QCoh(X)$ derived ∞-category of $\cup$| quasi-coherent sheaves of $\mathcal{O}_X$-modules.

$\mathcal{C}^\omega = Perf(X)$

$X \to Y$

$QCoh(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} QCoh(Y)$, & only f^* continuous (f_* not continuous)

Then, it turns out:

curved A_∞-deformations of $\mathcal{C}$ up to equivalence $\longleftrightarrow$ $HH^*(\mathcal{C})[1] \in \mathrm{dgLie}_k$ (obs. of Lowen)

(ex: Infinitesimal case: each 2-cycle of $HH^*(\mathcal{C})$ gives a curved deformation of $\mathcal{C}$ over $k[t]/t^2$.

Goal: avoid talking about curved deformations, just ordinary deformations.

Definitions:

$\mathrm{Alg}_k := \mathrm{Alg}(\mathrm{Mod}_k)$. There's a monoidal structure on algebras; ~~...~~ given by $\otimes_k$.

Then,

Thm [Lurie]: $\mathrm{Alg}(\mathrm{Alg}_k) \simeq \mathrm{Alg}_{E_2}(\mathrm{Mod}_k) =: \mathrm{Alg}^{(2)}_k$

Consider the functor: $\mathrm{Alg}_k \longrightarrow \mathcal{C}at^c_k$

$A \longmapsto L\mathrm{Mod}_A$ functoriality given by base change.

It turns out this functor is symmetric monoidal, so taking Alg, get

$$\underset{\mathrm{Alg}^{(2)}_k}{\overset{\shortparallel}{\mathrm{Alg}(\mathrm{Alg}_k)}} \longrightarrow \mathrm{Alg}(\mathcal{C}at^c_k)$$

$A \longmapsto L\mathrm{Mod}^{\otimes}_A$

(refinement/generalization of fact that modules over comm. alg. have a tensor product)

Hence,

If $A \in \mathrm{Alg}_k^{(2), \text{augmented}}$,

using fact LMod_A is $\otimes$.

$$\mathcal{C}at\,\mathrm{Def}_{\mathcal{C}}(A) := \{(\mathcal{C}_A, u),\ \mathcal{C}_A \in \mathcal{C}at_A^c,\ u : \mathcal{C}_A \otimes_A k \xrightarrow[\sim]{u} \mathcal{C}\}.$$

<u>Def</u> [Pridham, Lurie]: If $A \in \mathrm{Alg}_k^{(n)}$ is called <u>Artinian</u> if it satisfies

(1) $\pi_i A = 0 \quad \forall i < 0,\ i \gg 0$ (bounded above)

(2) $\dim_k \pi_i A < \infty$

(3) ~~$\pi_0 A$~~ $\quad r \subseteq \pi_0 A$; $\pi_0 A/r \simeq k$

radical = intersection of all maximal ideals / nilpotent elts. if $\pi_0 A$ fin-dimensional)

(i.e. analogue of "$\pi_0 A$ local Artinian w/ residue field k").

<u>Ex</u>: $k \oplus k[n]$ $n \geq 0$ shifted <u>square zero</u> extensions,

For any Artinian A, there is a filtration

$$A_n = A \to A_{n-1} \to \cdots \to A_0 \simeq k$$

with

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_{i-1} \arrow[d] \\ k \arrow[r] & k \oplus k[n] \end{tikzcd}$$

pullback

<u>Def</u>: [Lurie]: An E_n <u>formal moduli problem</u> $/k$ is a functor

$$F : \mathrm{Alg}_k^{(n), art} \longrightarrow \mathcal{S} \quad \leftarrow \infty\text{-cat. of spaces}$$

such that

(1) $F(k) \simeq *$

(2) $\forall$ pullback squares

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ k \arrow[r] & k \oplus k[n] \end{tikzcd}$$

in $\mathrm{Alg}_k^{n, art}$

Quillen

(<u>ex</u>:
$$\begin{tikzcd} k[t]/t^3 \arrow[r] \arrow[d] & k[t]/t^2 \arrow[d] \\ k \arrow[r] & k \oplus k[1] \end{tikzcd}$$

need replace by fibration ~~[illegible]~~ ~~[illegible]~~ (surjective map) for this to be a pullback, replace $k[t]/t^2$.

$$\begin{tikzcd} F(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(A) \arrow[d] \\ * \arrow[r] & F(k \oplus k[n]) \end{tikzcd}$$

is a pullback square.

<u>Def</u>: $\mathrm{FMP}_{E_n}(k) \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{(n), art}, \mathcal{S})$

"subcat. of formal moduli problems"

Thm [Lurie]: $\exists$ an equivalence of ∞-categories:

$$(*) \quad \begin{tikzcd} \mathrm{FMP}_{E_n}(k) \arrow[r, "\sim"] & \mathrm{Alg}_k^{(n), aug} \arrow[l, bend left, "\Psi"] \end{tikzcd} \ni B$$

augmented

(also for $n = +\infty$).

by "tangent complex" (or rather, augmentation ideal is tangent complex).

Informally:

$$\begin{tikzcd} k \oplus k[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k \arrow[d] \\ k \arrow[r] & k \oplus k[n] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(k \oplus k[n-1]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & F(k \oplus k[n]) \end{tikzcd}$$

$(T_F)_n = F(k \oplus k[n])$. Can arrange

$T_F \in \mathrm{Spectra}_k$

(Lurie, Pridham) $\mathrm{FMP}_{E_\infty}(k) \xrightarrow{T[-1]} \mathrm{dgLie}_k$

$F(k) \to F(A)$, ΩF, $T_{\Omega F} \simeq T_F[-1]$.
$F(k) = *$

<u>Problem</u>: the functor $A \mapsto \mathrm{CatDef}(A)$ is not a formal moduli problem. But, can naively force it to be via:

$\exists\, L = (-)^\wedge$ left adjoint (by <u>adjoint functor thm</u>; cats are presentable).

$$\mathrm{FMP}_{E_1}(k) \hookrightarrow \mathrm{Fun}_*(\mathrm{Alg}_k^{(1),\mathrm{art}}, \mathcal{S})$$

"pre-formal moduli problems".

$*$ {based, meaning $F(k) = *$}

Given $F \in \mathrm{Fun}_*(\mathrm{Alg}_k^{(1),\mathrm{art}}, \mathcal{S})$, unit of adjunction gives a natural map

$$F \longrightarrow F^\wedge.$$

If $\mathcal{C}$ is a k-linear ∞-category, have $\mathrm{CatDef}_{\mathcal{C}} : \mathrm{Alg}_k^{(2),\mathrm{art}} \longrightarrow \mathcal{S}$ is <u>not</u> a formal moduli problem, but as before there is a morphism

$$\mathrm{CatDef}_{\mathcal{C}} \xrightarrow{\Theta_{\mathcal{C}}} \mathrm{CatDef}_{\mathcal{C}}^\wedge.$$

<u>Thm</u> [Lurie] (DAG X): for any $A \in \mathrm{Alg}_k^{(2),\mathrm{art},\mathrm{aug}}$, $\exists$ eq. equiv.

$$\mathrm{CatDef}_{\mathcal{C}}^\wedge(A) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}\left(\mathcal{D}^{(2)}(A), HH^\bullet(\mathcal{C})\right)$$

the E_2-koszul dual of A ('b/c A augmented!)

using usual E_2-str. on $HH^\bullet(\mathcal{C}) = \mathrm{End}_{\mathrm{End}(\mathcal{C})}(\mathrm{id}_{\mathcal{C}})$

$:= \mathrm{End}_{k \otimes_A k}(k)$

<u>bar construction</u>

E_2 koszul <u>dual</u>. (Higher algebra)

Given $B \in \mathrm{Alg}_k^{(1),\mathrm{aug}}$, $\Psi_B(A) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{(1)}}(\mathcal{D}^{(1)}(A), B)$

E_1-koszul dual.

gives the formal deformation problem? (where Ψ is image in Lurie's thm, $(*)$)

<u>Ex</u>: $\pi_0 \mathrm{CatDef} \xrightarrow{(**)} HH^2(\mathcal{C})$

$$\left(\mathcal{D}^{(2)}(k[t]/t^2) \simeq \mathrm{Free}_{E_2}(k[-2])\right)$$

Ex: $\mathcal{C} = \mathrm{Mod}$

$B = k[u, u^{-1}]$

$|u| = 2$. (there exist cocycles in HH^2 not in image of $(**)$; so map not an equivalence)

Thm (Lurie): If $\mathcal{C}$ is bounded above, meaning $\left(\mathrm{Ext}^n_{\mathcal{C}}(C, C') = 0 \ \forall \ n >> 0 \right)$ "stably cptly generated"?
(not necessarily uniform)

then $\Theta_{\mathcal{C}}$ induces an isomorphism for $i \geq 1$,
and is injective on π_0.

Conjecture: if $\mathcal{C}$ is smooth and proper, then $\Theta_{\mathcal{C}}$ is an equivalence.

$k[[t]] = \lim_{i \geq 1} k[t]/t^i$, & extending to pro-Artinian objects by general nonsense, define $F(k[[t]]) = \lim_i F(k[t]/t^i)$

Thm: (B.-Katzarkov-Pandit) Let $\mathcal{C}$ be a bounded above k-linear ∞ category which has a compact generator, then

$$\mathrm{CatDef}_{\mathcal{C}}(k[[t]]) \xrightarrow{\sim} \mathrm{CatDef}^{\wedge}_{\mathcal{C}}(k[[t]])$$ is an equivalence. (claim may not work over $\mathbb{Z}/2$ grading)

If $\{\mathcal{C}_i\}_{i \geq 1}$ is a formal deformation of $\mathcal{C}$, then each $\mathcal{C}_i$ has a compact generator.

(if $\mathcal{C}$ is proper, each $\mathcal{C}_i$ is also proper, but smoothness of $\mathcal{C}$ doesn't imply $\mathcal{C}_i$ smooth)

the problem here is algebraizability of deformations: $\mathcal{C}_i$ existing $\not\Rightarrow$ $\exists$ deformation algebraic over $k[[t]]$ unless $\mathcal{C}$ is proper.

Define $\mathrm{Aut}_{\mathcal{C}} = \Omega \mathrm{CatDef}_{\mathcal{C}}$ (deformations of the identity of $\mathcal{C}$ in endomorphisms out of $\mathcal{C} \otimes A$ = bimodules over) (so def of ~~bimodules~~; generally only get indcoherent bimodules)

and $\mathrm{Aut}^!_{\mathcal{C}} = \Omega \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$

space of indcoherent bimodules

Prop: $\forall \ \mathcal{C}, \exists \ \mathrm{CatDef}^{\wedge}_{\mathcal{C}}(A) \simeq \mathrm{Map}_{Gp(FMP)}(\Omega \mathrm{Spf}(A), \mathrm{Aut}^!_{\mathcal{C}})$

Lem: $FMP \xrightarrow[\sim]{\Omega} Gp(FMP)$; taking loops gives directly the equivalence.

Scheme of proof of thm: start w/ a cpct generator $\langle E \rangle = \mathcal{C}$. Then,

$$\mathcal{D}^{(2)}_f(k[[t]]) \simeq k[\beta_2]$$
$$|\beta_2| = 2.$$

(Take image of β) get $k[\beta_2] \to HH^*(\mathcal{C})$ classifying deformations (note if $\mathbb{Z}_2$ graded ϕ_E could $= \mathrm{id}$, problem)

$\downarrow$

$\beta \mapsto (-) \otimes \mathrm{Ext}^*(E, E)$, gives map $E \xrightarrow{\phi_E} E[2] \longrightarrow E^\beta = \mathrm{Cone}(\phi_E)$. call this obstruction

& claim E^β is a new cpct gen. over $\mathcal{C}$, fixed under gp.