Y. Groman, <u>Wrapped Floer theory and Lag'n torus fibrations</u> 1/29/2017

Goals: 1) X symp. manifold, $f: X \to \mathbb{R}^n$ Lag'n torus fibration, maybe w/ singularities.
want to define $SH^*(X;f)$, $HW(L_i, L_j)$, closed-open operators.
Rmk: in general, X is not conical at ∞

2) $X = \underset{\text{toric CY}}{M} \setminus \underset{\text{anticanon. divisor}}{D}$, e.g., $(M = \mathbb{C}^3, K_{\mathbb{P}^2}, -)$, $D = \{z_1 z_2 z_3 = 1\}$, carries a sLag torus fibration $f: X \to \mathbb{R}$ (Gross, Auroux)

<u>Thm</u>: A Lag'n section $s: \mathbb{R}^3 \to X$ satisfies Abouzaid's generation criterion, e.g.,

$$\mathcal{OC}: HH_{-n}(L) \to SH_0(X;f) \text{ hits } 1,$$

$\Rightarrow$ HMS of X with $\mathrm{Spec}\, HW^0(L,L)$

§1. Let X be semipositive (helps control neg. chern hol. spheres), (e.g., $c_1 \geq 0$).
$f: X \to \mathbb{R}^n$ a Lag'n torus fibration, and $h: \mathbb{R}^n \to \mathbb{R}$, J almost cplx. structure. ($J \in \mathrm{End}(TX)$, $J^2 = -\mathrm{Id}$)

Q i) For which (h, J) is $HF^*(h\circ f, J)$ well-defined?

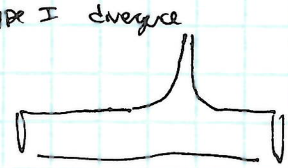
ii) For (h_1, J_1), (h_2, J_2) s.t. $h_2 \geq h_1$, when do we have a canonical map $HF^0(h_1 \circ f, J_1) \to HF^0(h_2 \circ f, J_2)$?

(motivation: relate HMS to SYZ by using Hamiltonians pulled back from base of fibration.)

Issue: compactness for moduli spaces. (need "contractible condition" to get e.g., continuation maps)

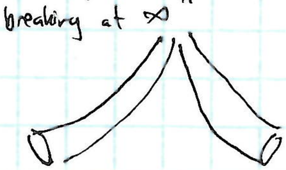
Schematically, two types of "bad" divergent behavior for sequences of trajectories

Type I divergence



(no control over diameter on cpct. subsets)
(relatively easy to deal with)

Type II divergence
"breaking at ∞"



Condition I: need bounded geometry of the metric g on (H, J)

$$X_\psi \times \mathbb{R} \quad \text{determined by} \quad \tilde{\omega} = \omega + dH \wedge dt + ds \wedge dt$$

mapping torus of time-1 flow, ψ, of H

and

$$J_H = \begin{pmatrix} J & 0 \\ (X_H)^{0,1}? & j \end{pmatrix}$$

(where $X_\psi = X \times [0,1] / (p,0) \sim (\psi(p), 1)$)

(i.e., means g is complete, sectional curvature bdd. above $\sec \leq K$,
injectivity radius $\mathrm{inj} > \varepsilon$.

This rules out type 1 divergence. (uses some sort of isoperimetric inequality/monotonicity)

Say have $u: \mathbb{R} \times S^1 \to X$ & $K \subseteq \mathbb{R} \times S^1$ cpct. subset, then want:

(*) $\mathrm{diam}(u(K)) \leq E(u) + \mathrm{area}(K)$ (so for any cpct. subset, have control of diameter.

<u>Rmk</u>: This condition can be weakened to a contractible condition s.t. we still have (*).

Condition II; ~~[scribbled out]~~:

$$d(p,\psi(p)) > \varepsilon \text{ on } X \setminus K, \quad K \text{ a cpct. set.} \quad (\text{here, } \psi \text{ is the time 1 flow})$$

Rules out type II divergence for u of finite energy, e.g., when

$$E(u) := \int_S \underbrace{\| \partial_t u - X_{H_t} \|^2_{L_2}}_{A(s)} \, ds < E < \infty$$

Why? $A(s) \ll \varepsilon$ outside a set of finite measure in $\mathbb{R}$.

$\Rightarrow d(x,\psi(x)) < \varepsilon$. $\Rightarrow$ for most s, the loop $u(s,-) \cap K \neq \phi$.

(now use fact that we've ruled out type I divergence)

<u>How to verify these conditions in practice</u>: have $f: X \to \mathbb{R}^n$, $h: \mathbb{R}^n \to \mathbb{R}$.

Pick: $g: U \subseteq X \longrightarrow \mathbb{R}^n$ local action coordinates; meaning

$\left(\exists\, \Theta: U \setminus \text{sing fibers} \longrightarrow \mathbb{T}^n \text{ s.t. } X_{g_i} = \dfrac{\partial}{\partial \theta_i}\right)$.

Let $h \circ f = \tilde{h} \circ g$. Then require

(condition II'): $\| \nabla \tilde{h} - \underset{\substack{\cap \\ \mathbb{Z}^n}}{n} \| \geq \varepsilon$ on $\mathbb{R}^n \setminus K$; ("non-integrability condition" for a flow along a torus)

integral vector (corresponds to: "slope at ∞ is not in the period spectrum" for a Liouville domain.)

Now, define $SH^*(X;f) = \varinjlim_{(h,J) \text{ condition I + II'}} HF^*(h \circ f, J)$

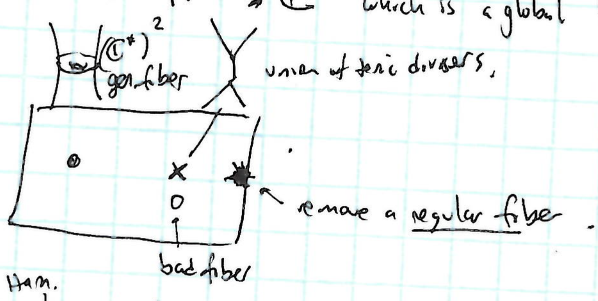
<u>Rmk</u>: when (h,J) satisfy conditions, we have well-defined HF's & well-defined continuation maps. when $h_1 \leq h_2$.

Interesting case: M = Toric CY, D anticanonical divisor.

Fact: • $M = \mathbb{C}^{n+3} /\!\!/ G$, $G \subseteq T^{n+3}$ subtorus preserving the volume form $dz_1 \wedge \cdots \wedge dz_{n+3}$, i.e., preserves $\prod_{i=1}^{n+3} z_i =: P$

• P induces a function $P: M \to \mathbb{C}$ which is a global holomorphic function.

picture:

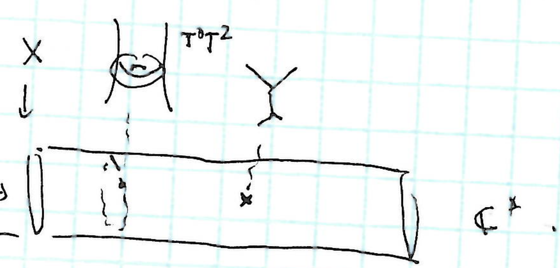


• There is a 2-dim. Ham. torus action on M preserving the fibers of P, call its moment map μ.

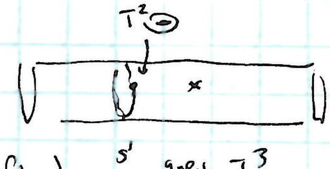
$\mu: (\mu_1, \mu_2): M \to \mathbb{R}^2$

• Consider $X = M \setminus \{P^{-1}(1)\}$, P induces:

(have to modify ~~ω~~ ω on X to retain bounded geometry: "inflate ω" (by pulling back stk. from base — canonical).



Rmk: Get a Lag'n T^3 fibration on X via $\left(Re(Log(1-p)), \mu_2, \mu_3\right)$.



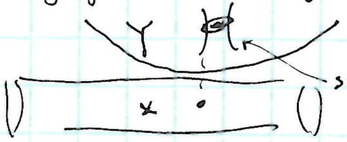
Now, work w/ a Hamiltonian of the form $H_F + H_B$, where $H_F = h_F \circ \mu$ (flows in fibers), goes T^3.
and $H_B = h\left(\underbrace{Re(Log(1-p))}_{\pi}\right)$. (choose h generic so critical pts. don't occur at crit-value of π).

Computing SH:

- contractible periodic orbits: • periodic orbits over the critical point of H $\{dh=0\}$: P_1
  - critical points of π: $d\pi = 0$: P_2.

computation: $\deg(P_2) \geq 2$, so to compute SH^0, only need to worry about P_1.

Concavity-type argument: Floer trajectories connecting orbits in 1-fibre stay close to the fibre.



so trajectories must stay near fibre => no differential!

$$\Rightarrow SH^{0,0}(X;f) = \mathbb{C}[u^{\pm1}, v^{\pm1}].$$

0^{th} contractible part "$SH^0(T^*(S^1)^2)$"

Similar arguments establish $SH^{0,\geq 0}$ (wind positive) $= \mathbb{C}[x, u^{\pm1}, v^{\pm1}]$, & $SH^{0,\leq 0} = \mathbb{C}[y, u^{\pm1}, v^{\pm1}]$.

and

$$\Rightarrow SH^0(X;f) = \mathbb{C}[x,y,u^{\pm1},v^{\pm1}] \Big/ xy = F(u,v)$$

something contractible, make no claims about this fcn currently.

Let $S :=$ Lag'n section

claim 1: x, y are in the image of the open-closed map. $\mathcal{OC}$.

claim 2: $HH_{-n}(L)$ is a free SH^0 module of rank 1. (+n in talk)

(1+2) => Let Ω be a generator. Then, $\mathcal{OC}(\Omega)$ is a unit.