M. Porta, Derived analytic geometry 1/27/2017 Miami.

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1. Motivations

DAG is useful for:

* better control on intersection theory (e.g., Serre's intersection formula)
* better control on deformation theory (formal moduli problems)
* DAG reveals hidden structures:
  - ↳ derived Brauer group
  - ↳ shifted symplectic structures.

Same motivations for DAnG. There's more:

* Griffiths period map $\left(\begin{array}{c} X \\ \downarrow f \\ S \end{array} \right) \longmapsto$ variations of pure Hodge structure) (requires analytic topology)

Done by J. Holstein and C.D. Natale

* (in progress) Rieman-Hilbert correspondence

X smooth over $\mathbb{C}$ $\left\{ \begin{array}{c} \text{Local systems on } X \times S \\ \text{relative to } S \end{array} \right\} \overset{\text{analytic iso.}}{\simeq} \left\{ \begin{array}{c} \text{Coh. sheaves on } X \times S \text{ with an} \\ \mathcal{O}_S\text{-linear flat connection} \end{array} \right\}$

(S not necessarily smooth: can one allow S derived? yes (in progress))

Joint work with Tony Yue Yu

Broad goal: use derived geometry to build GW-invariants for non-archimedean spaces.

Need: $\exists$ derived Deligne-Mumford analytic stack $\mathbb{R}\overline{M}_{g,n}(X,\beta)$ s.t.

(i) $t_0(\mathbb{R}\overline{M}_{g,n}(X,\beta))$ is classifying stable maps

(ii) quasi-smooth: $\mathbb{L}^{an}_{\mathbb{R}\overline{M}_{g,n}(X,\beta)}$ perfect and in amplitude $[-1,0]$

"analytic cotangent complex"

* A good framework for dAnG well adapted to the non-arch. setting
* A tool to re-organize derived DM analytic stacks.

2. Overview

Def: A derived scheme is a pair $(X, \mathcal{O}_X)$; X top. space, $\mathcal{O}_X$ is a sheaf of simplicial commutative algebras. (or, bounded above cdgas, ok in char. 0). s.t.

(1) $(X, \pi_0 \mathcal{O}_X)$ is a classical scheme

(2) $\pi_i(\mathcal{O}_X)$ are quasi-coherent as sheaves of $\pi_0(\mathcal{O}_X)$ modules.

Classically, analytic spaces

$An_{\mathbb{C}} :=$ full subcat. of locally ringed spaces.

(First attempt to define derived analytic space: replace "scheme" by $An_{\mathbb{C}}$ in (1) & drop "quasi-coherent" requirement in (2) (b/c no such notion)
but this doesn't work! why?

Rmk: Taking $dAn_{\mathbb{C}}$ to be a full subcat. of locally simplicially ringed spaces would give a wrong answer.

Key issue with such a definition, $\mathbb{L}^{an}_{\mathbb{A}^1_{\mathbb{C}}}$ would be infinite dimensional.

Problem: too many derivations!
(Recall: a derivation is a $\mathbb{C}$-linear map $A \xrightarrow{d} M$ (alg. module) with $d(ab) = a\,db + b\,d(a)$. (inductively apply Leibniz)
$\Rightarrow$ for any polynomial $f(a)$, $d(f(a)) = f'(a)\,da$)
(but for a random convergent power series, this doesn't force anything! so there may be many derivations w/ different values on e.g., e^x).
(~~want~~ need: the derivation d should be "continuous" in a sense, so ~~can~~ can work w/ $f(z) = \sum \frac{1}{n!} z^n$)
((could: work w/ simplicial Banach algebras; but they don't behave well from a categorical perspective; would need instead to work with "Ind-Banach" algebras
instead: axiomatize the properties of a Banach algebra that are important in this setting))

One way: simplicial Banach algebras $\rightsquigarrow$ problems at categorical level ($\nexists$ arbitrary colimits?)
Alternate sol'n (Lurie):

$a \in A$ Banach algebra comm/$\mathbb{C}$. i can think of a as

$$\forall U \subset \mathbb{C} \qquad \begin{tikzcd} \mathbb{C}[z] \arrow[r, "f_a"] \arrow[d] & A \\ \mathrm{Hol}(U) \arrow[ur, dashed, "\exists\ \text{lift}"'] & \end{tikzcd}$$

Solution: if $\sigma_a =$ spectrum of a, then the lift exists iff $\sigma_a \subseteq U$.

Define $T_{an}(\mathbb{C}) = \{ U^{open} \subseteq \mathbb{C}^n$ + holomorphic maps between them (do not respect embedding in $\mathbb{C}^n$) (no higher ~~[illegible]~~ morphisms) $\}$.

& consider:

$$F : T_{an}(\mathbb{C}) \longrightarrow Set / \mathcal{S} \quad (= sSet \text{ or } Spaces)$$

$$\begin{array}{ccc} \mathbb{C} & \longmapsto & F(\mathbb{C}) \\ \cup\! \mid & & \cup\! \mid \\ U & \longmapsto & F(U) \end{array}$$

should be a ring.

Def: An analytic ring is a functor $A : T_{an}(\mathbb{C}) \longrightarrow Spaces$ s.t.
(1) A commutes w/ products
(2) A commutes with pull backs
$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd}$$
where this is an open immersion.

This gives ring structure via:

$$A(\mathbb{C}^2) = A(\mathbb{C}) \times A(\mathbb{C})$$
$$\downarrow A(+)$$
$$A(\mathbb{C})$$

but also gives $A(\mathbb{C}) \xrightarrow{A(\exp)} A(\mathbb{C})$.

this is an axiomatization of "holomorphic function calculus."

In the rigid analytic setting, simply need to modify $T_{an}(\mathbb{C})$: say K is a non-archimedean field: Then,

$$T_{an}(K) = \left\{ \begin{array}{l} \text{smooth } K\text{-analytic spaces that are} \\ \text{separated, paracompact, strict} \end{array} \right\}$$

(or, could work with affinoid domains, but more problematic b/c cannot evaluate at $\mathbb{A}^1$)

(corresp. could replace $T_{an}(\mathbb{C})$ by just subsets of ball, & result wouldn't change!)

~~3. Main Results~~

Def: A derived analytic space is a pair $(X, \mathcal{O}_X)$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of simplicial analytic rings such that

(1) $(X, \pi_0(\mathcal{O}_X)^{alg})$ analytic space$/\mathbb{C}$.

means replace F by algebra $F(\mathbb{C})$

(2) $\pi_i(\mathcal{O}_X)^{alg}$ are coherent

In nonarchimedean setting same choice works; one needs to be ~~more~~ slightly careful about choice of X.

Ex: X a complex manifold.

(b/c $T_{an}(\mathbb{C}) \subseteq$ cplx manifolds)

Then, $A = \mathrm{Hom}(X, -): T_{an}(\mathbb{C}) \longrightarrow \mathrm{Set}$ satisfies all the relevant axioms, hence gives dAn space

(gives embedding $An_{\mathbb{C}} \hookrightarrow dAn_{\mathbb{C}}$).

Thm (Lurie, P-Yu):

(1) $\exists$ ∞-cat. of $dAn_{\mathbb{C}} / dAn_K$ (non-arch.)

(2) This category admits fiber products.

(3) $An_{\mathbb{C}} \xhookrightarrow[\text{fully faithful}]{} dAn_{\mathbb{C}}$, $An_K \xhookrightarrow{f.f.} dAn_K$.

$\exists$ derived analytification functor

$$(-)^{an} : dSch^{afp} \longrightarrow dAn_{\mathbb{C}} / dAn_K \quad \text{s.t.}$$

"almost of finite presentation (means $\pi_i \mathcal{O}_X$ coherent)

(1) $\forall\ (Y, \mathcal{O}_Y) \in dAn_{\mathbb{C}}$, there is an equivalence

$$\mathrm{Map}_{dAn_{\mathbb{C}}}\left((Y,\mathcal{O}_Y), (X,\mathcal{O}_X)^{an}\right) \simeq \mathrm{Map}_{\mathrm{Top}}\left((Y,\mathcal{O}_Y^{alg}), (X,\mathcal{O}_X)\right)$$

$\uparrow$ simplicially ringed spaces

(2) The map $(X^{an}, \mathcal{O}^{alg}_{X^{an}}) \longrightarrow (X, \mathcal{O}_X)$ is flat in the derived sense, e.g.,

$$\pi_i(\mathcal{O}_{X^{an}}) = \pi_i(\mathcal{O}_X) \otimes_{\pi_0(\mathcal{O}_X)} \pi_0(\mathcal{O}_{X^{an}}).$$

<u>Thm</u> (P):

(1) If $f: X \longrightarrow Y$ is a proper map of $dSch^{afp}$, then the diagram

$$\begin{tikzcd} Coh^+(X) \arrow[r, "(-)^{an}"] \arrow[d, "Rf_*"'] \arrow[dr, phantom, "\circlearrowleft"] & Coh^+(X^{an}) \arrow[d, "Rf_*^{an}"] \\ Coh^+(Y) \arrow[r, "(-)^{an}"] & Coh^+(Y^{an}) \end{tikzcd}$$

<u>Rmk</u>: (could work w/ unbounded cat, & result also holds for Artin stacks, where + is unavoidable; for schemes, can remove +)

(2) If X is proper ~~dSch~~ $dSch^{afp}$ — unbounded derived category.

$$\underline{Coh}(X) \xrightarrow{\sim} \underline{Coh}(X^{an})$$

<u>Def. theory</u>:

<u>Thm</u>: $(P)_{cpx}$:

(1) $\exists\ \mathbb{L}^{an}_{X/Y}\quad \forall\ f: X \longrightarrow Y \in dAn_{\mathbb{C}}$

(2) All standard properties ~~of~~ of cotangent cplx are satisfied (e.g., behavior w.r.t. fiber products)

(3) If $f: X \to Y$ is a closed immersion, then

$$\mathbb{L}^{an}_{X/Y} \simeq \mathbb{L}_{X^{alg}/Y^{alg}}.$$

(2, 3, 4 give algorithm to compute $\mathbb{L}^{an}$; locally spaces admit closed immersions to $\mathbb{A}^n$, where 4 is known).

(4) If $X \in dSch^{afp}$, $(\mathbb{L}_X)^{an} \simeq \mathbb{L}^{an}_{X^{an}}$.

Thm: (P., P.-Y.) :

Let $F : d\,\mathrm{Stn}^{op} \longrightarrow \mathcal{S}$ be a sheaf for the analytic topology. Then, TFAE:

↑ derived Stein or affinoid
(derived analytic space whose underlying classical space is Stein/affinoid)

(1) $\exists$ a derived analytic space $(X, \mathcal{O}_X)$ $\in dAn_{\mathbb{C}}$ which represents F.

(2) F satisfies the following conditions:

(i) $t_0(F) = F \circ j$, $j : \mathrm{Stn} \hookrightarrow d\mathrm{Stn}$
is representable

(ii) F admits a global analytic cotangent complex

(iii) F behaves well on square-zero extensions and postnikov towers.

^ e.g., if $\mathcal{U} = \mathrm{colim}_n\, t_{\leq n} \mathcal{U}$,
then $F(\mathcal{U}) = \lim F(t_{\leq n} \mathcal{U})$.