T. Spaide, CY structures, spherical functors, & shifted symplectic structures 1/28/2017 w/ L. Katzarkov & P. Pandit
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Structures on categories —
CY structures
Motivation: Y^d a smooth projective scheme, then $\mathrm{Perf}(Y)$ has a Serre functor $S_Y = - \otimes K_Y[d]$.
If Y is Calabi-Yau, then $S_Y \simeq [d]$.

More generally, let $\mathcal{Y}$ be a proper category.
Naive idea: A (d-)CY structure is an isomorphism $S_{\mathcal{Y}} \simeq [d]$.
$\Longleftrightarrow$ a map $\omega : HH_\bullet(\mathcal{Y}) \longrightarrow k[-d]$ (k: char 0 alg. closed field)
with some non-degeneracy: for all $X, Y \in \mathrm{ob}\,\mathcal{Y}$, the pairing

$$\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,X) \longrightarrow \mathrm{Hom}(X,X) \longrightarrow HH_\bullet(\mathcal{Y}) \xrightarrow{\omega} k[-d]$$ is non-degenerate.

Def: A weak right CY structure on $\mathcal{Y}$ is an ω as above.
This is not sufficient for many applications, e.g., constructing TQFTs.
Def: A right CY structure on $\mathcal{Y}$ is $\tilde{\omega} : \underset{HC(\mathcal{Y})}{\overset{\shortparallel}{HH(\mathcal{Y})_{S^1}}} \longrightarrow k[-d]$

such that the composition $HH_\bullet(\mathcal{Y}) \to HH_\bullet(\mathcal{Y})_{S^1} \xrightarrow{\tilde{\omega}} k[-d]$ is non-degenerate as above,
e.g., is a weak right CY structure.

Ex: 1) Y smooth proper CY, then $\mathrm{perf}(Y)$ has a right CY structure
2) [Ganatra]: Y ~~compact~~ symplectic manifold,
$\mathrm{Fuk}^{\mathrm{cpct}}(Y)$ has a right CY structure.

Spherical functors:
Def: Say $F : \mathcal{X} \to \mathcal{Y}$ is a functor. F is spherical if it has a right adjoint $F^!$ and a left adjoint F^* such that
- $\mathrm{Cone}(\mathrm{id}_{\mathcal{X}} \to F^!F) =: T$ is an equivalence, and
- $F^! \longrightarrow F^!FF^* \longrightarrow TF^*$ is an equivalence.

Say $\mathcal{Y}$ has a d-CY structure, and $F : \mathcal{X} \to \mathcal{Y}$ is spherical.
By general theory, $F^! \simeq S_{\mathcal{X}} F^* S_{\mathcal{Y}}^{-1}$

$$\overset{\shortparallel}{TF^*} \simeq \overset{\shortparallel}{S_{\mathcal{X}}[d]}\, F^*$$ ← b/c $\mathcal{Y}$ is CY-d.

F is compatible with CY structure if $\exists$ iso $T \simeq S_{\mathcal{X}}[-d]$ such that the above commutes.

Ex: 1) X Fano scheme,

$Y \overset{i}{\subset} X$ smooth anticanonical. By adjunction, Y is CY and

$\mathrm{Perf}(X) \xrightarrow{i^*} \mathrm{Perf}(Y)$ is compatible spherical.

2) Y a d CY scheme,

$X \overset{j}{\subset} Y$ a smooth divisor,

$\mathrm{perf}(X) \xrightarrow{j_*} \mathrm{Perf}(Y)$ is compatible spherical.

3) Abouzaid-Ganatra-Seidel:

X
$\downarrow w$ LG model with compact critical locus and, say, exact fibers; call Y the generic fiber.
$\mathbb{C}$

Then, $\underset{\text{fibrewise cpct}}{FS^c(X,w)} \xrightarrow{\cap} \underset{\text{cpct. Lagns.}}{F^c(Y)}$ is compatible spherical.

Relative CY structures

Def: Let $\mathcal{X}$, $\mathcal{Y}$ be categories; ω is a weak CY structure on $\mathcal{Y}$, and $F: \mathcal{X} \to \mathcal{Y}$ a functor.
Then, a (right) weak relative CY structure on F is a homotopy $F \circ \omega \simeq 0$, where

$$F \circ \omega : HH_\cdot(\mathcal{X}) \xrightarrow{F} HH_\cdot(\mathcal{Y}) \xrightarrow{\omega} k$$; satisfying a non-degeneracy condition:

For all $X, Y \in \mathcal{X}$,

$$\underbrace{\mathrm{Hom}_{\mathcal{X}}(X,Y)}_{\text{term 1}} \to \overbrace{\mathrm{Hom}_{\mathcal{Y}}(FX,FY) \underset{\substack{\text{weak}\\ \text{CY}}}{\simeq} \mathrm{Hom}_{\mathcal{Y}}(FY,FX)^*[-d]}^{\text{term 2}} \xrightarrow{F^*} \overbrace{\mathrm{Hom}_{\mathcal{X}}(Y,X)^*[-d]}^{\text{term 3}}$$

[1]

is a fiber sequence e.g.)

Given a right CY structure $\tilde{\omega}$ on $\mathcal{Y}$, a (right) rel. CY structure on $F: \mathcal{X} \to \mathcal{Y}$ is a homotopy $F \circ \tilde{\omega} \simeq 0$ ~~such~~ ~~that~~ inducing as before a weak rel. right CY structure (e.g., non-degeneracy holds)

Thm: (Katzarkov-Pandit-Spaide): Let $F: \mathcal{X} \to \mathcal{Y}$ ~~be~~ a functor w/ left and right adjoints, ~~then~~ and ω a weak CY structure on $\mathcal{Y}$. Then, F is compatible spherical if and only if it's weak relative CY.

(in general, can't always upgrade this to (strong) rel. CY,) (Missing additional structure on LHS!)

Analogy: weak CY structure — $S_{\mathcal{Y}} \simeq [d]$

rel. CY structure — F is compatible spherical.

Q: what additional structure should we have to get a full relative CY structure?

Ex: i) $\underset{\text{Fano}}{\mathrm{Perf}(X)} \xrightarrow{i^*} \underset{\text{anticanon}}{\mathrm{Perf}(Y)}$ has a full relative CY structure [Calaque].

• $X \xrightarrow{w} \mathbb{C}$, Y general fiber, &

$FS^c(X,w) \xrightarrow{\cap} \mathrm{Fuk}^c(Y)$ should have a relative CY structure as follows:

$$\begin{array}{ccccccc} HH_\bullet(FS^c(X,w)) & \longrightarrow & HH_\bullet(\mathrm{Fuk}^c(Y)) & & & & \\ \downarrow{\scriptstyle \text{Seidel}} & & \downarrow{\scriptstyle \text{Ganatra}(S^1\text{-eq.})} & & pt & \longmapsto & 1 \\ H_{n+1-*}(X, X_\infty) & \longrightarrow & H_{n-*}(Y) & \longrightarrow & H_0(Y)[-n] & \longrightarrow & \mathbb{C} \\ \downarrow & & & & & & \\ \alpha \in H_1(X, X_\infty) & & \longrightarrow & & \partial\alpha. & & \end{array}$$

$\alpha = \sum c_i \alpha_i$

If S^1-equivariance exists, $\cap$ is right relative CY.

Shifted symplectic structures: [PTVV] (a closed n-shifted 2-form, + non-degeneracy)

If Y is a derived stack, there is a notion of an n-shifted symplectic structure on it.

If $X \xrightarrow{f} Y$ is a morphism, have the notion of a Lagrangian structure on f.

There is a moduli of objects functor: Rmk: to get covariant functor, rather than contravariant, map → need to stick to saturated categories. (full)

$dgCat \xrightarrow{\mathcal{M}_\bullet}$ derived stacks [Toën - Vaquié], and a d-CY structure on a category $\mathcal{Y}$ gives us a $(2-d)$-shifted symplectic structure on $\mathcal{M}_{\mathcal{Y}}$.

(n.b., a weak CY structure gives a non-deg. 2-form which is not closed.),

• A relative CY structure on $f: \mathcal{X} \longrightarrow \mathcal{Y}$ gives a Lag'n structure on

$\mathcal{M}_f : \mathcal{M}_{\mathcal{X}} \longrightarrow \mathcal{M}_{\mathcal{Y}}$. [Brav-Dyckerhoff].

Ex: If Y is a smooth scheme, a 0-shifted symplectic structure on Y is the same as a symplectic structure in the usual sense, &

$X \hookrightarrow Y$ is Lag'n iff the inclusion $i: X \to Y$ has a Lagrangian structure (uniquely determined)

• Perf BG have 2-shifted structures.

• If Y has n-shifted structure, X is d-CY, then

$\mathrm{Maps}(X, Y)$ has an $(n-d)$-shifted structure.

Thm: (Katzarkov - Pandit - Spaide): If Y is dCY, and $X \subset Y$ smooth divisor, then

$j_*: \mathcal{M}_{\mathrm{Perf}(X)} \longrightarrow \mathcal{M}_{\mathrm{Perf}(Y)}$ has a Lagrangian structure (haven't checked it comes from a relative CY structure).

Thm: (Pantev–Toën–Vaquié–Vezzosi): Say Y has an n-shifted symplectic structure, and

$$\begin{array}{c} X_1 \searrow \\ Y \\ X_2 \nearrow \end{array}$$

have Lagrangian structures.

Then, $X_1 \times_Y X_2$ has an $(n-1)$ shifted symplectic structure.

Ex: Let W be a closed oriented 3-manifold, and let $X \subset W$ be a closed oriented 2-manifold, splitting W into W_+, W_- with $\partial W_+ = \partial W_- = X$.

X is a "2-Calabi-Yau" (actually it's not, but $\exists$ an idea of "orientation structures" on derived stacks," one idea is for a CY variety),

so $\mathrm{Map}(X, BG)$ has a 0-shifted sympl. structure.

And $\mathrm{Maps}(W_\pm, BG) \xrightarrow{\text{restr.}} \mathrm{Map}(X, BG)$ have a Lag'n structure on them.
"Fano" "anticanonical"

" $\pm \# \, Loc_G(W_+) \times_{Loc_G(X)} Loc_G(W_-)$ is the Casson invariant of W. "

Actually: take the algebraic intersection # of these; or instead, by PTVV: the product $Loc_G(W_+) \times_{Log_G(X)} Log_G(W_-)$ has a (-1)-shifted symplectic structure.

Behrend $\Rightarrow$ can take a virtual count of such things (using Behrend fcn + VFC).

Hope: we can attach spherical functors together in this way.

Rmk: If X has a (-1) shifted structure, then there is a quasi-iso. $\mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[-1]$ (tangent cplx., cotangent cplx.)

$\Rightarrow \chi(\mathbb{T}_X) = 0$ "expected dimension $= 0$" (should then give a symmetric obstruction theory)

Pantev: Q. ("quantize in direction of -1 structure, get a sheaf of cplexes on intersection; taking χ get Behrend function").