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Based on discussions w/ T. Kragh, A. Blumberg.
(Floer htpy of Lagr.)

M closed symp.
$\cup$
L Lag'r.

$\boxed{\text{Fix a basepoint}}$

$\Omega^2(M,L)$ space of based discs in M w/ ∂ on L.

$\pi_0\, \Omega^2(M,L) = \pi_2(M,L) \leftarrow$ group (concatenation)

$\boxed{x \mid y}$ ...

In fact $\Omega^2(M,L)$ is a group in top. spaces. (little annoying to make mult. associative)
(A_∞ space - ~)

$\Rightarrow$ Suspension spectrum (i.e., stable htpy) is an A_∞ ring
($\Rightarrow$ all ~~gen~~ homology theories have coherent multiplication)

Idea: Moduli spaces of hol. discs will deform this A_∞ structure.

$\to$ first, have to introduce the correct "Thom space": we consider the index bundle of the linearized $\bar\partial$ operator as a (stable) vector bundle on $\Omega^2(M,L) \xrightarrow{\text{Thom space}} \boxed{\Omega^2(M,L)^\nu}$ $\leftarrow$ Spectrum.

(to e.g. deal w/ issue that product is typically not $\mathbb{Z}$ graded, etc.)

Key point: pre-Gluing for ker/coker of $\bar\partial$ operator $\Rightarrow$ this vector bundle is multiplicative, under:

$$\Omega^2(M,L) \times \Omega^2(M,L) \longrightarrow \Omega^2(M,L)$$

($\Rightarrow$ Thom spectra are multiplicative)

① Abstract htopy. theory

② Relation to homology w/ local coeffs.

Paths $* \to L$ in M. ($* \in L$)

$\mathcal{P}_{*,L} M$

① Have: $M \xrightarrow{\text{tangent classifying}} BU$

monodromy of vec. bdle

$$\begin{array}{ccccccccccc}
 & & & & \text{Have:} & \to & \Omega(M,L) = \mathcal{P}_{*,L}M \xrightarrow{\text{fiber}} & L & \longrightarrow & M \\
\Omega^2(M,L) & \to & \Omega(L) & \to & \Omega M & & & & & \\
\downarrow & & \downarrow & & \downarrow \text{"monodromy"} & & \downarrow (*) & \downarrow & & \downarrow \\
\Omega(U/O) & \to & O & \to & U & \to & U/O & \to BO & \longrightarrow & BU
\end{array}$$

this map classifies index bundle; & is the loop Ω of (*)

$\Omega(U/O) = \mathbb{Z} \times BO$

U/O: Lag'n Grassmannian.

$\Rightarrow$ multiplicative,
hence the Thom space is multiplicative.

② It's familiar to take a local system on a top. space $X \rightsquigarrow H_*(X, \nu)$ (II)

& this can be realized as the homology of some Thom spectrum

If: $X \times X \xrightarrow{m} X$ (associative)

& local system has a "product" (e.g., $\nu \otimes \nu \to m^* \nu$), then (II) is a ring too.

(So far, just htpy theory: only requires L "almost/formal Lag'n").

Conj: (M is symplectically aspherical):

$\Omega^2(M,L)^\nu$ admits a curved A_∞ deformation induced by moduli spaces of hol. curves, as in [FOOO]'s book, meaning e.g.:

① Invariance under Symp.

② Ham. isotopies induce Morita equivalences after localization. (Novikov-type localization)

In homology: A curved A_∞ algebra (assuming e.g., curvature lives in some higher energy) filtered & has a category of modules. (genuine dg cat.)

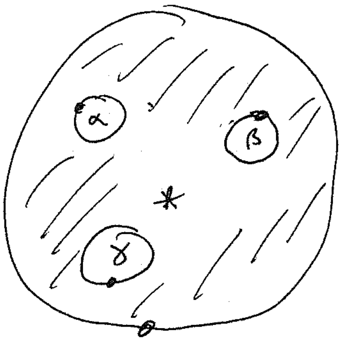
Morita equivalence means: For any pair (L, L'), can form a bimodule, which induces an equivalence of module categories if $L \overset{Ham}{\sim} L'$

(it's clear there's a categorical structure too, but more complicated).

Rmks: No topological assumptions on L (Spin, orientable, etc.) b/c ~~b~~ we twist by ν.

Can analyze the obstruction to trivializing ν at a given prime p.

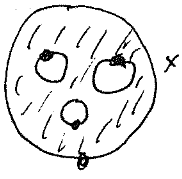
$\rightsquigarrow$ for each such (M, L), the Floer htopy type of L is well-defined (as a curved A_∞ alg) after inverting a finite # of primes.



In general (M not aspherical), have to go one step back:

$$\begin{array}{ccccc} \Omega^2 M & \longrightarrow & \Omega^2(M,L) & \longrightarrow & \text{—} \\ \downarrow & & \downarrow & & \\ \mathbb{Z} \times BU & \longrightarrow & \mathbb{Z} \times BO & & \end{array}$$

First consider $\overset{\text{Map}(S^2,M)}{\Omega^2 M} = \text{Map}((D^2, \partial D^2), (M, *))$ as a framed E_2 algebra. (framed little discs).

e.g., operad:  $\times (\Omega^2 M)^3 \longrightarrow \Omega^2 M.$ 

$\alpha, \beta, \gamma \longmapsto$

The map $\Omega^2 M \longrightarrow \Omega^2(M,L)$

$\quad\circlearrowright fE_2 \qquad\qquad \circlearrowright A\infty$ ✓ Index bundle of $\bar\partial$ operator on spheres

makes $\Omega^2(M,L)^\vee$ into a ring "over" $\Omega^2 M^\vee$

e.g., $\left(\alpha\cdot(x\cdot y) \simeq (\alpha x)\cdot y \atop {\cong \atop x\cdot(\alpha y)} \right)$.

Conjecture: The moduli space of spheres determines a curved framed E_2 deformation of $\Omega^2 M^\vee$..

(no need to use $\mathbb{Q}$-coeffs, b/c using framings)

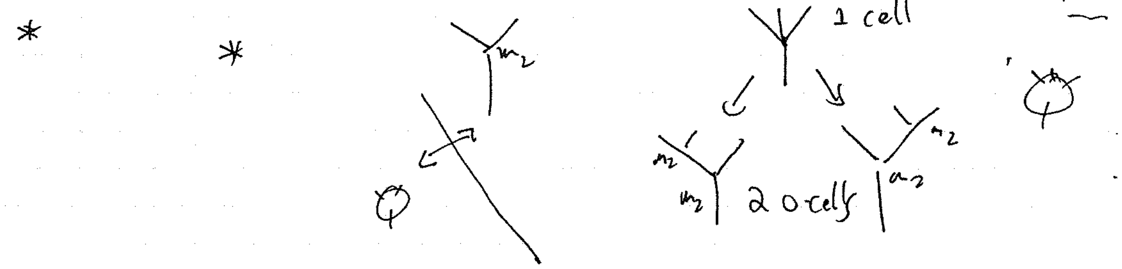
The space of discs defines a (curved) algebra over $\Omega^2 M^\vee$.

The aspherical case:

Recall we can model $A\infty$ algebras using the Stasheff operad, with its cellular decomposition labeled by trees.

deg 0   deg 1

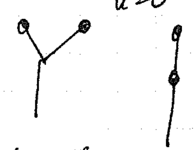
For $k=0$ $k=1$ $k=2$ $k=3$ $k=4$



Curved case: "Add unstable discs."

But, if you just form the operad by adding, at $k=0$ $k=1$

then formally enlarge:



& higher things

$\rightsquigarrow$ get an acyclic operad. (i.e., homology is 0!

$\Rightarrow$ "curved operad & not homology of an operad in spaces" can make sense of this as an operad in stacks or spectra).

In hol. curve theory, have a little more:

(γ) $\gamma \in \pi_2(M,L)_+$ ✓ has a positive homotopy class.

So now, reformulate:

Let Γ be a (discrete) monoid (assume commutative) (e.g., $\pi_2(M,L)_+$ "non-neg. area htpy classes of discs" $\cup$ Γ ← classes with a non-empty moduli of stable discs (for some fixed J) or rather think about $\Gamma \subset H_2(M,L)$ (commutative), or rather e.

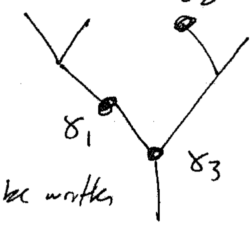
Defn Γ is gapped if, $\forall \gamma \in \Gamma$ $\exists$ only finitely many solns to the equation $\sum \gamma_i = \gamma$ with $\gamma_i \neq 0$.

(Gromov cpctness $\Rightarrow$ Γ is gapped)

Define the Γ-graded curved A_∞ operad to be:
built from stable trees ~~labeled~~ w/ vertices labeled by Γ.

stable means: if have univalent/bivalent vertices, they need to have $\gamma \neq 0$.
(c.f. Behrend-Manin in GW theory).

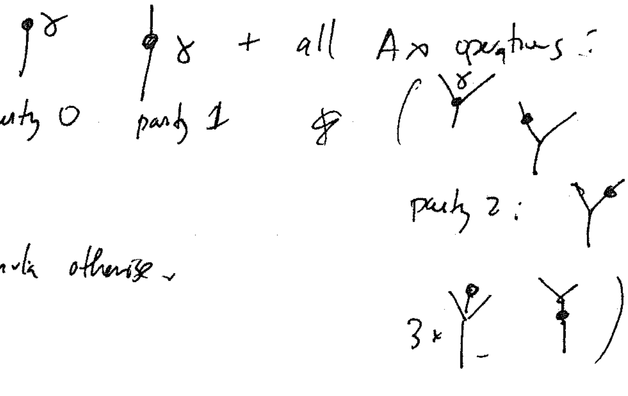
This "operad" $\mathcal{O}_\Gamma$ decomposes as
$$\mathcal{O}_\Gamma \cong \coprod_{\gamma \in \Gamma} \mathcal{O}_\gamma \longleftarrow \mathcal{O}_0$$
$\mathcal{O}_0$: the usual A_∞ operad.
$\mathcal{O}_\gamma$: trees whose total sum is γ.

(e.g., 

$\in \mathcal{O}_{\gamma_1+\gamma_2+\gamma_3}$)

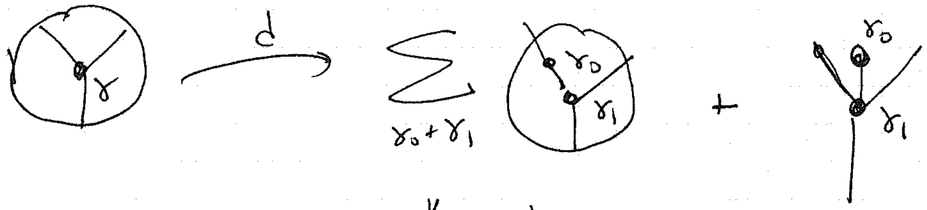
There's a filtration on $\mathcal{O}_\Gamma$ by ("# ways γ can be written as a sum (non-degenerate) γ's")

If γ is minimal (i.e., no $\gamma_1+\gamma_2=\gamma$), only have



Why Γ gapped? Hard to build a formula otherwise.

for instance want



this may be an ∞ sum..

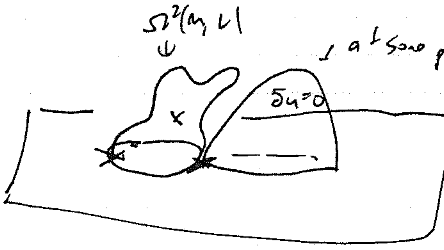
Then, (In particular, the gapped version is not acyclic).

Nov. ring $\Lambda_0 \cong \mathbb{Z}[\pi_2(M,L)_+]$

↑ completed. (but in this formalism, completion doesn't show up at this stage,) b/c keeping track of π_2 gradings.

Conj: $\Omega^2(M,L)_+^{\vee}$ (pos. area part of $\Omega^2(M,L)$) acquires the structure of an algebra over $\mathcal{O}_\Gamma$.

- The curvature μ_0 is the "count" of moduli spaces of hol. discs passing through the basepoint.
- The differential "μ_1" is "string bracket" (in space of discs, not loops) w/ moduli spaces of discs w/ 1 marked point (which don't need to go through basepoint)



, at some point, (intersects) runs through ~~a basepoint~~ a family of discs through basepoint (at another time) → concatenate to get another elt. of $\Omega^2(M,L)$

Can form the category of modules over this operad. Then how to localize?

$$\text{mod-}\mathbb{C}[[t]] \xrightarrow{?} \text{mod-}\mathbb{C}[[t,t^{-1}]].$$

$t\to 0$ ↑ ~~supp~~

mod $\mathbb{C}$ ← localize along this subcategory.

Q: what is the homology of $\mathcal{O}_\Gamma$.

Now, emulate here: get a map 0-area maps

$$\Omega^2(M,L)_+ \to \Omega^2(M,L)_0$$

⇝ localize by killing this module.

conj: this achieves the desired result.

↑ this is an algebra over, even over $\mathcal{O}_\Gamma$ (w/ γ's acting by 0)

Rmk: none of this requires moduli spaces to be smooth.