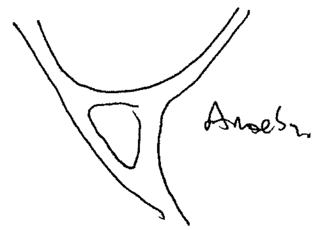
D. Auroux, Speculations on mirror symm. for affine hypersurfaces

$$(\mathbb{C}^*)^n \supset H = f^{-1}(0)$$

$$f(x_1,\dots,x_n) = \sum_{\substack{\alpha\in A\\ \text{finite} \subseteq \mathbb{Z}^n}} \tau^{\rho(\alpha)} x^\alpha \qquad \tau\to 0 \quad \rho: A\to\mathbb{R} \text{ convex,}$$

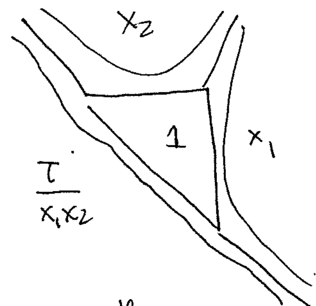
ρ extends to a piecewise affine convex fxn. on Conv(A) convex hull; determines polyhedral decomp $\mathcal{P}$ of Conv(A).

$$H \xrightarrow[\operatorname{Log}_{1/\tau}]{} \mathbb{R}^n \qquad \text{im}:$$



approaches $\to$ $\operatorname{Trop}(f)(\xi_1,\dots,\xi_n) = \max_{\alpha\in A} (\langle \xi,\alpha\rangle - \rho(\alpha))$.

Ex: $f = x_1 + x_2 + \frac{\tau}{x_1x_2} + 1$



Wrapped Fukaya cat. of H $\mathcal{W}(H)$

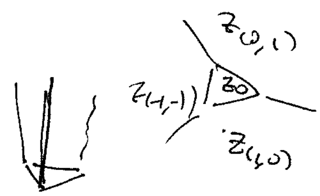
$\xleftrightarrow[\text{HMS}]{}$ der. cat. of sheaves on mirror, or rather

$$D^b_{sg}(Z) := D^b\mathrm{Coh}(Z)/\mathrm{perf}(Z).$$

[conj. 1d: H. Lee
Gammage - Shende
+GPS]

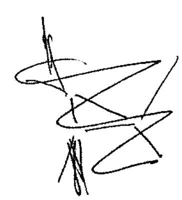
To get Z: define

$$\Delta_Y = \{(\xi_1,\dots,\xi_n,\eta)\in\mathbb{R}^{n+1} / \eta \geq \operatorname{Trop} f(\xi_1,\dots,\xi_n)\}.$$



Y = toric var. corresp. to this

$Y = \mathcal{O}(-3)$
$\downarrow$
$\mathbb{P}^2$.



nc toric CY
(smoothing of mirror of a maximal degeneration)

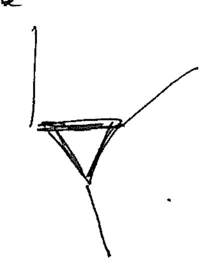
$W = z^{(0,\dots,0,1)}$ = toric monomial vanishing to order 1 on all facets.

$Z = W^{-1}(0) = \bigcup$ toric divisors in Y.

Rmk: under an assumption (*), $D^b_{sg}(Z) = D^b\mathrm{Coh}(D_0)$ [Orlov]
but may not be

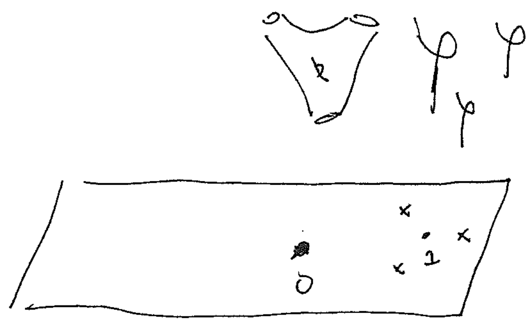
(e.g. that there's one facet)

$$D_0 = Z_0 \cap \bigcup_{\alpha\neq 0} Z_\alpha.$$



(*) $\Leftrightarrow$ Y ...

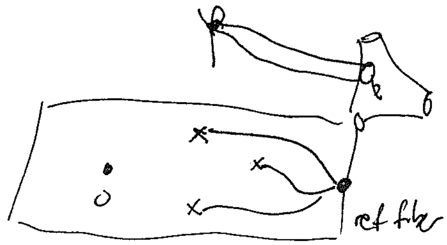
$$\begin{array}{c}(\mathbb{C}^*)^n\\ \big\downarrow f\\ \mathbb{C}\end{array}$$



3 sing. fibers (nodal ell. curves)

Fukaya–Seidel cat. $\mathcal{F}((\mathbb{C}^*)^n, f)$: objects are Lag'ns in $(\mathbb{C}^*)^n$ w/ bd. in ref. fiber of f.

(e.g., $f^{-1}(0)$, $f^{-1}(2)$)

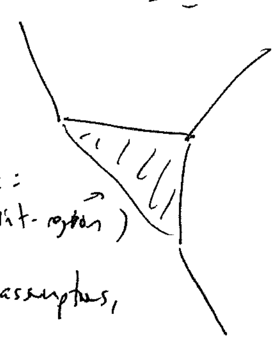


$\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ full subcat. "supported in region where constant term dominates"

[cf. Abouzaid sections of Log map over a region?].

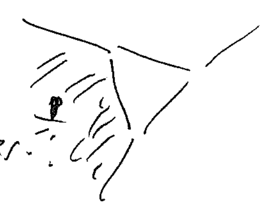
Under assumption (*) (0 ∈ every max. cell of $\mathcal{P}$), they're the same.

(~~too~~ agree w/ thimbles in these simple case: one int. region)



[Abouzaid's thesis]: $\mathcal{F}^\circ((\mathbb{C}^*)^n, f) \simeq D^b\mathrm{Coh}(Z_0)$ (under some assumptions, assumption (*) for instance; or rather may get an embedding $\hookrightarrow$).

Rmk: constant component need not be bounded;
so then $\mathcal{F}^\circ$ will have non-cpct Lag'ns, "wrap in the fiber."

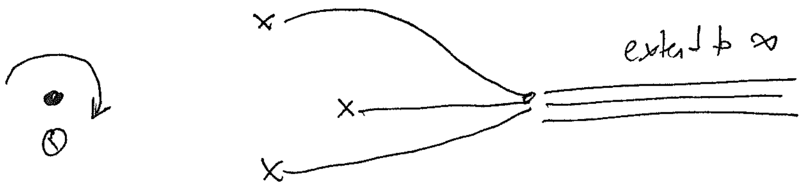


Assertions:

- ∃ two types of acceleration functors,
$$\alpha_0 \;\&\; \alpha_\infty : \mathcal{F}((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n - \overset{f^{-1}(0)}{\overset{\shortparallel}{H}}).$$
- ∃ functor $\overset{\text{restr.}}{\rho} : \mathcal{W}((\mathbb{C}^*)^n - H) \longrightarrow \mathcal{W}(H)$

  lifting $j : \mathcal{W}(H) \longrightarrow \mathcal{W}((\mathbb{C}^*)^n - H)$
- ∃ dist. triangle $\alpha_\infty \longrightarrow \alpha_0 \longrightarrow j\rho\,\alpha_0 \longrightarrow \alpha_\infty[1]$.

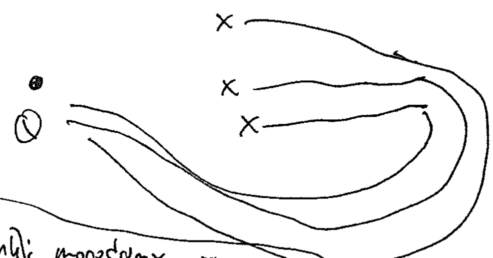
α_∞:



extend to ∞ & realize as actual filtered subcomplex of $\mathcal{W}((\mathbb{C}^*)^n - H)$.

(depends on extension to ∞: canonical if $(*)$ $\Leftrightarrow$ $\nexists$ other crit. points.

α_0: turn clockwise & extend to 0 & do same thing:



Rmk: $(\mathbb{C}^*)^n \setminus H$ is a Liouville ~~domain~~ m'fold b/c

(1) Take $\theta_{(\mathbb{C}^*)^n} + d^c(\log|f|)^2$

or (2) $(\mathbb{C}^*)^n \setminus H = \{(x_1, \dots, x_n, y) \in (\mathbb{C}^*)^{n+1} \,/\, y + f(x_i) = 0\}$.

[Rmk]: monodromy around 0 is id., even though monodromy around ∞ is not.

(this is a ~~side~~ game one can play

What's ρ? there's a general construction, when monodromy around H is trivial, from $\mathcal{W}((\mathbb{C}^*)^n \setminus H) \to \mathcal{W}(H)$ ($\mathbb{Z}/2$-graded).

Obs 1:

$$\begin{tikzcd}
\mathcal{W}((\mathbb{C}^*)^n \setminus H) \arrow[r, "\text{(Ab-Se restrict)}"'] \arrow[rr, bend right, "\text{OR}"'] & \mathcal{W}(\mathbb{C}^* \times H) \arrow[r, "\text{``project''}"] & \mathcal{W}(H)
\end{tikzcd}$$

$\uparrow$ nbhd of H

(defined to send arc $\times L \longmapsto L$, & extend).

$\exists$ $\Theta \in SH^2((\mathbb{C}^*)^n \setminus H)$ gives a natural transformation, via $\mathcal{CO}$:

(orbits going around H once) $\qquad \mathrm{id} \longrightarrow \text{monodromy} = \mathrm{id}[2]$

Then, localization w.r.t. this.

$L_1, L_2 \in \mathcal{W}((\mathbb{C}^*)^n \setminus H)$, then

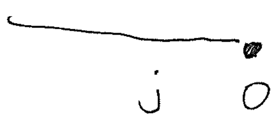
$HW^*_H(\rho(L_1), \rho(L_2)) \cong_k HW^{i+2k}(L_1, L_2)$, where are connects by

$$\begin{tikzcd} W((\mathbb{C}^*)^n - H) \arrow[r, "\rho"] \arrow[d, leftrightarrow, "i"] \arrow[dr, phantom, "\circlearrowright \text{conj.}"] & W(H) \arrow[d, leftrightarrow, "j"] \\ D^b Coh(Z) \arrow[r, "\text{quotient}"'] & D^b_{sg}(Z) = D^b Coh(Z)/Perf(\end{tikzcd}$$

$$hom^i_{D^b_{sg}}([\mathcal{E}_1], [\mathcal{E}_2]) \cong \varinjlim Ext^{i+2k}_{D^b Coh}(\mathcal{E}_1, \mathcal{E}_2),$$

which matches above guess.

The lifting j transports:



& the exact triangle $\leftrightarrow$ counting sections

$$\begin{tikzcd} F((\mathbb{C}^*)^n, f) \arrow[r, bend left, "\alpha_0"] \arrow[r, bend right, "\alpha_\infty"'] \arrow[dd, leftrightarrow, bend right] & W((\mathbb{C}^*)^n \setminus H) \arrow[r, "\rho"] & W(H) \arrow[dd, leftrightarrow, bend left, "\text{conj.}"] \\ & & \\ D^b Coh(Z_0) \arrow[r, bend left, "i_*"] \arrow[r, bend right, "-\pi^*"'] & D^b Coh(Z) \arrow[r, "loc"] & D^b_{sg}(H) = D^b Coh(D_0) \end{tikzcd}$$

note: $\rho\,\alpha_0 = \partial$ restr. functor [Abouzaid-Seidel];

$\rho\,\alpha_\infty = 0$

i

always works (i_*)

requires (*) to hold to get a projection $Z \mapsto Z_0$ ($-\pi^*$)

otherwise not canonical

<u>Ex:</u> If $f = x_1 + \cdots + x_n + 1$, then

$H = \mathcal{P}_{n-1}$ $(n-1)$-dim'l pair of pants

& $(\mathbb{C}^*)^n \setminus H = \mathcal{P}_n$ n-dim'l pair of pants.