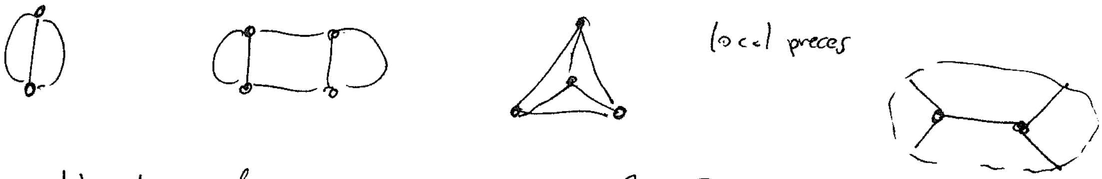
R. Casals, Morse flow trees in graph Legendrians

(arXiv: 1705.01034 [CM] + in progress) $|V| = 2g+2$, $|E| = 3g+3$, $|F| = g+3$.

G a cubic planar graph, $G = (V, E, F)$



Aim: study Leg. surfaces

$$\Lambda_G \overset{\text{def.}}{\subseteq} J^1 S^2 \subseteq \mathbb{R}^5$$

depends on a knot $S^2 \subset \mathbb{R}^5$.

satellite assoc. to some knot.

The algebra $\mathcal{A}_G$ of binary sequences on G is the free graded ~~algebra~~ $\mathbb{F}[e_1^{\pm 1}, \dots, e_{3g+3}^{\pm 1}]$-algebra

generated by $\{f_1, \dots, f_{g+3}\}$ and $\{x_{11}, x_{10}, x_{00}\}$; with $|f_i| = 1$ and $|x_{ij}| = 2$.

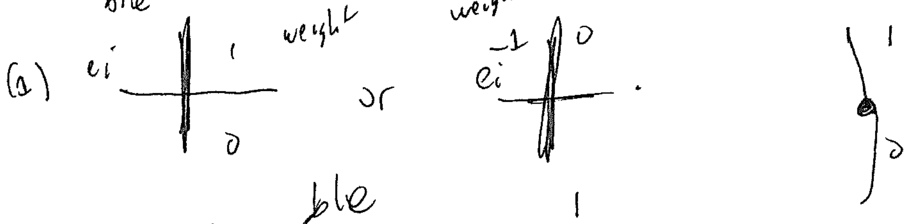
Define $\partial : \mathcal{A}_G \to \mathcal{A}_G$ by $\partial f_i = \sum_{v \in f_i} w(v)$, where $w(v) = e_\ell e_m e_n^{-1}$.

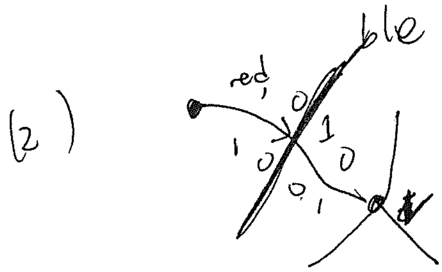
if

f_i, e_ℓ, e_n, e_m, v.

Given a decorated G, a binary sequence along γ_f is a function

$B(\gamma_f) : \gamma_f \longrightarrow \{0, 1\}$ with rules?





when ~~these~~ crossed, can switch or not.

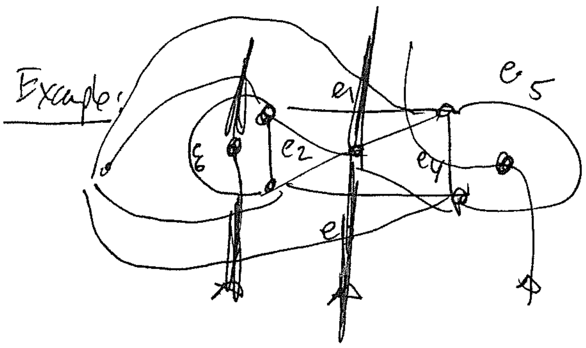
with ~~...~~ weight $w(v)$ if using it

$\rightsquigarrow$ Laurent monomial: every time cross edge, multiply by weight.

Then, a binary seq $B(\gamma_f)$ has a monomial, the weight

$$w(B(\gamma_f)).$$

Example:



$$\partial f_1 = e_2 e_6 (e_1^{-1} + e_3^{-1})$$

$$\partial f_2 = e_2 e_6^{-1} (e_1 + e_3) + e_4 e_5^{-1} (e_1 + e_3)$$

$$\partial f_3 = e_4 e_5 (e_1^{-1} + e_3^{-1}).$$

$$\partial x_{11} = \sum_{f \in F} \sum_{B \in B(\gamma_f)} w(B), \qquad = (\quad) f_1 + (\quad) f_2 + (\quad)$$

counts binary sequences from 1 → 1

1-term.

2-terms: whether we used red path or not

↯

<u>Thm</u>: ([CM]): The pair (A_G, ∂_G) defines a dg algebra, i.e., $\partial_G^2 = 0$, & its isomorphism type (but not its true iso-type) is independent of choices (of blue + red paths).

In addition, given a tree T spanning all vertices $V \setminus \{v_0\}$, we can base change to $\mathbb{F}[e_1^{\pm 1}, \dots, e_{2g}^{\pm}]$ where $e_1, \dots, e_{2g}$ are the edges on T (by changing variables), (by sending some e_i's that which are not in tree to 1.)

and obtain (A_T, ∂_T) a dg-algebra s.t.

$$\left| \mathrm{Aug}(A_T, \partial_T)(\mathbb{F}_q) \right| = \chi^{G^*}_{chrom}(q+1)$$

(incl. appendix by (K.-Sackel?)).

1-dim'l reps

since taking dual G^*, # of colorings of faces.

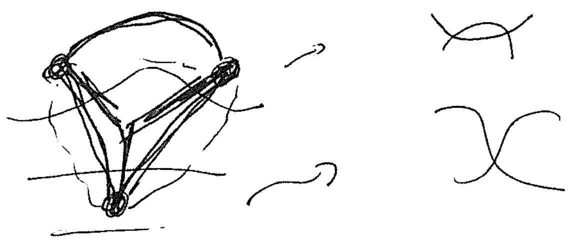
II. Legendrian topology :

Consider the holomorphic $(J^1\mathbb{C},\ \alpha = dz - w\,du)$, the holomorphic cusp

$\gamma(\tau) = \cancel{(\overset{u}{\tau^2}, \overset{w}{\tau}, \overset{z}{\tau^3})}$ $\quad (\overset{u}{\tau^2},\ \overset{w}{\tfrac{3}{2}\tau},\ \overset{z}{\tau^3})$.

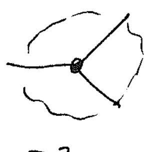
By considering real parts (of z), we get a Legendrian disk in $J^1\mathbb{R}^2$, with front

D_4^-

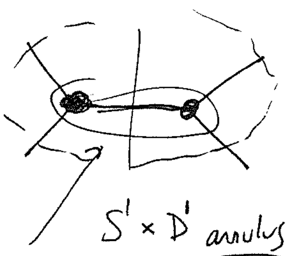


Then, this D_4^- Leg. front gives a recipe to build a Legendrian surface $\Lambda_G \subseteq J^1 S^2$ by changing vertices to D_4^- fronts.

Ex:

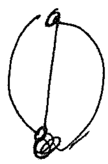


D^2.



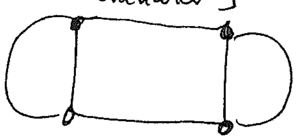
$S^1 \times D^1$ annulus.

H_1 class



is a leg. S^2

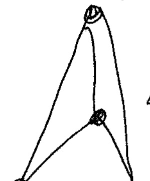
[Chekanov]



torus

&

[Clifford],

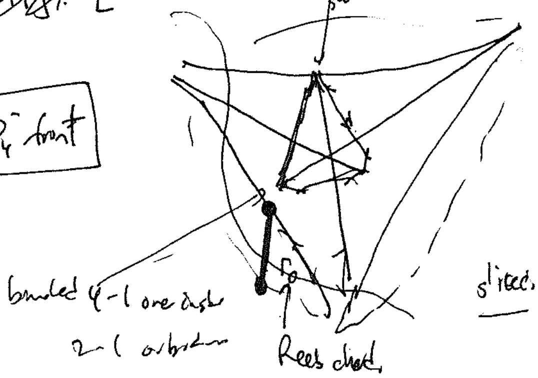


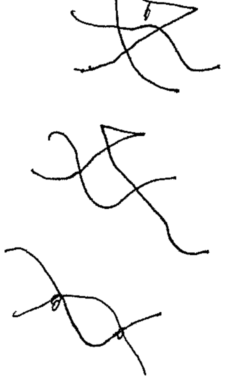
& generally, genus g (if use a G w/ $2g+2$ vertices. ~~edges~~)

are both tori,

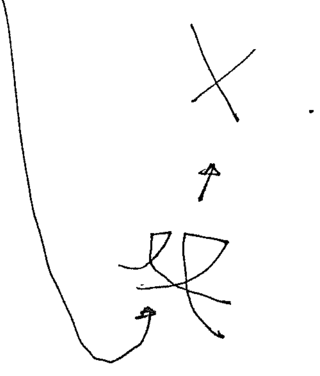
In order to study Floer theory of Λ_G, we ~~~~ first perturb D_4^- to a generic front + apply ~~~~. [

$\boxed{\text{generic } D_4^- \text{ front}}$





slices.



Technical Lemma:

$\exists$ a unique rigid tree with boundary condition ~~the following~~ on the v_0 path.

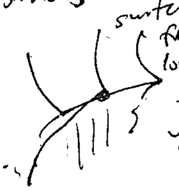
Proof: flow tree, pos. puncture

$$\dim(\Gamma) = -2 + I_{Morse}(\Gamma^+) - \sum \left(I_{Morse}(\Gamma^-) - 1 \right) + e(\Gamma) - s(\Gamma) - \underbrace{Y_1(\Gamma)}_{=0}$$

(# Y_1 singularities?)

"end" (only for switches) # times tree ends on cusp.

"switches" switch from lower to upper



(Global construction: take a Strebel differential whose foliation is the graph, & take associate spectral curve — gives a legendrian).

Thm: (1) The dg-alg. (A_T, ∂_T) is the leg. dga of $\Lambda_G \subseteq \mathbb{R}^5$. (unknot)

In particular, degree 0,1 parts are the dga of $\Lambda_G \subseteq J^1 S^2$.

(2) $\mathrm{Aug}(A_T, \partial_T) \cong Sh_{\Lambda_G}$ [Treumann–Zaslow], & counting objects, both #'s agree w/ colorings. ($\cong$ as sets, $\cong$ as sets (colorings))

Q. (Q: if put Novikov Q' variable in, what invt. of G does one get?!

III. Cluster geometry:

- The edges give a $(\mathbb{C}^\times)^{3g+3}$ chart (w/ a natural intersection pairing), which is Poisson given by intersection pairing in homology.
- The tree T gives a symplectic leaf $(\mathbb{C}^\times)^{2g}$.

In particular, Aug $\subseteq (\mathbb{C}^\times)^{2g}$ hol. Lag'n.

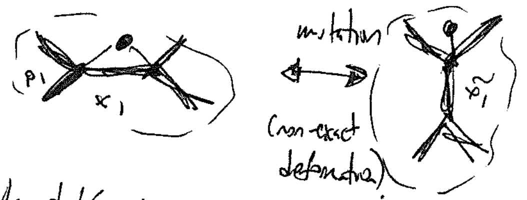
dim $2g - (g+3) + 3$
faces, $PSL_2(\mathbb{C})$.

Thm: (1) for each g, different graphs glue as cluster charts, w/ sympl. structure.

(2) The augmentation variety defines a Lagrangian.

Proof: (1) Look at how a graph changes fixing g:

only move:



$\longleftrightarrow$ non-exact deformation.

& locally compute differential difference for q:

$d(\circ) = x_1(1+p_1^{-1})$ $\qquad$ $d(\circ) = \tilde{x}_1$ $\qquad$ $\longleftrightarrow$ mutation.

(2) Compute for one graph & apply moves in (1).

In fact, can be upgraded to K_2 symp. form. (something Motivic..
(sympl. structure on))

(in fact, it's an exact Lag'n, & its primitive is an open GW invt. of a non-exact filling (after maybe a conifold transition)).