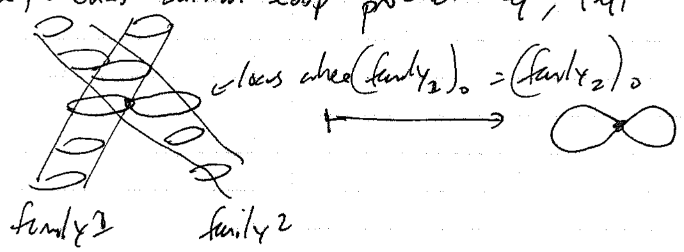
K. Cieliebak, Poincaré duality for free loop spaces

w/ N. Hingston, A. Oancea in progress

① Puzzles in string topology :

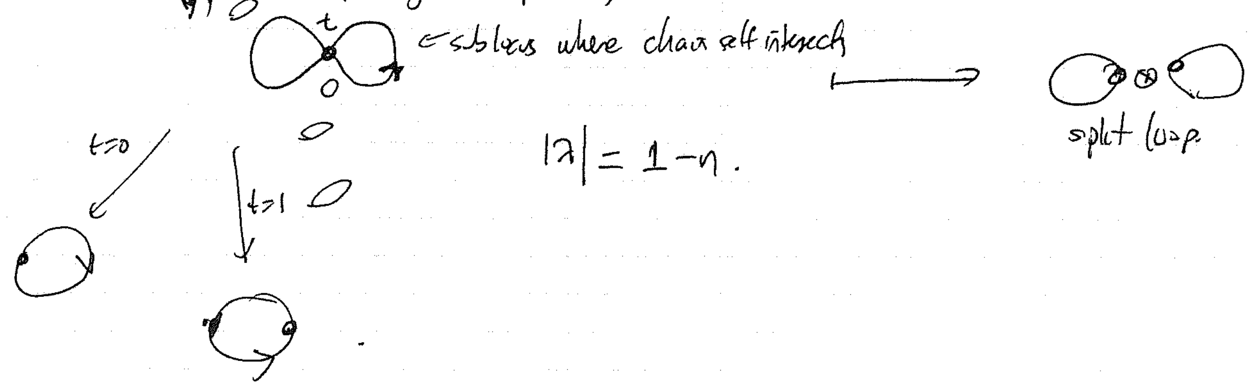
Q^n closed oriented manifold, $LQ = C^\infty(S^1, Q)$ free loop space.

$H_*(LQ)$: Chas-Sullivan loop product. μ, $|\mu| = -n$.



(intersection to this locus requires chains to be transverse)

On $H_*(LQ)$: Goresky-Hingston coproduct, λ.



$|\lambda| = 1-n$.

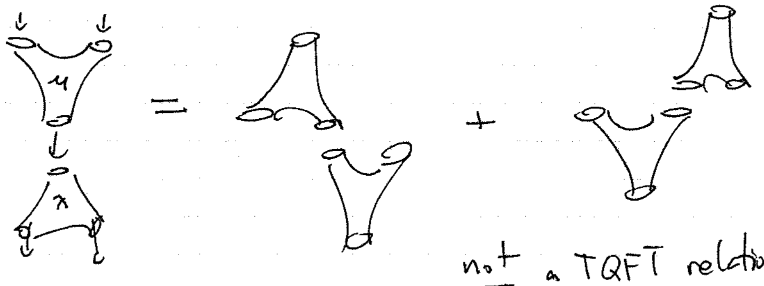
Questions

(a) dual statements $\mu \longleftrightarrow \lambda$?

(b) degrees are off?

(c) can we leave constants in?

(d)

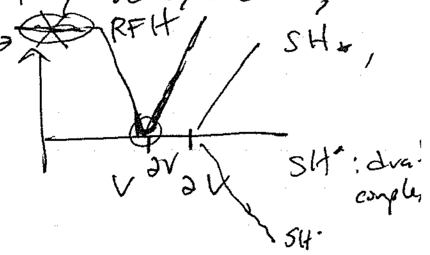


not a TQFT relation.

(More analogous to a Lie bialgebra relation — but graded μ no symmetric, not skew-symmetric)

② SH & RFH

V^{2n} Liouville domain.



where?

these orbits drop out action-wise (b/c H large).

Then, Thm: [C-Frauenfelder-O.] $\exists$ an LES:

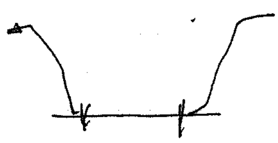
$$\cdots \to SH^{-*}(V) \to SH_*(V) \to RFH_*(\partial V) \to SH^{1-*}(V) \to \cdots$$

↑ hol. curves depend on filling, but generators only depend on V.

Here, one might think of

$RFH_*(\partial V)$ as "$SH_*(\partial V \times [0,1])$" $\neq$ "$SH_*(\partial V)$".

when SH_*(cobordism) is



& might think of $SH^{-*}(V)$
$$\begin{array}{c} \| \\ SH_*(V, \partial V) \end{array}$$

In this convention, $SH_0(V)$ is a unital ring.

RMK: nobody has worked out the SFT model of RFH. & exact sequence

constant loop part of exact sequence (action = 0):

$$\begin{tikzcd} \cdots \arrow[r] & SH^{-*}(V) \arrow[r, "\varepsilon_*"] \arrow[d] & SH_*(V) \arrow[r, "\text{ring homomorphism}"'] & RFH_*(\partial V) \arrow[r] & \cdots \\ & H^{n-*}(V,\partial V) \arrow[r, "\varepsilon_*"'] & H^{n-*}(V) \arrow[u] \arrow[r] & H^{n-*}(\partial V) \arrow[r] & \cdots \end{tikzcd}$$

($SH_*(V)$: unital ring; $RFH_*(\partial V)$: unital ring)

The map ε_* factors through constant loops, hence:

$\rightsquigarrow RFH_* \underset{\substack{\text{up to}\\ \text{finite dim'l}\\ \text{stuff}}}{\approx} SH_* \oplus SH^{1-*}$.

Specialize to the case $V = D^*Q$, $\partial V = S^*Q$. This becomes:

($M := Q$ in case it switches later)

$$H^{-*}(LQ) \xrightarrow{\varepsilon_*} \underbrace{H_*(LQ)}_{} \to RFH_*(S^*Q) \to \cdots$$

μ Chas-Sullivan product (Abbondandolo-Schwarz) $|\mu| = -n$

m product (deg $-n$)

hasn't been prev. defined in string topology.

m induces a product on $SH^{1-*} = H^{1-*}(LQ, Q)$; dualizing, get right degree to be the Goresky-Hingston coproduct.

[expect/conj.], subring of RFH_* on chain level.

$\exists$ Goresky-Hingston coproduct.

③ ~~PD~~ PD for RFH

Thm: $\exists$ canonical iso. $RFH_*(\partial V) \xrightarrow[\cong]{\text{P.D.}} RFH^{1-*}(\partial V)$ whose constant part

$H^{n-*}(\partial V) \xrightarrow{\cong} H_{n-*-1}(\partial V)$

$(\partial V)^{2n-1}$

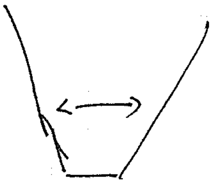
(recall $\boxed{\dim \partial V = 2n-1}$)

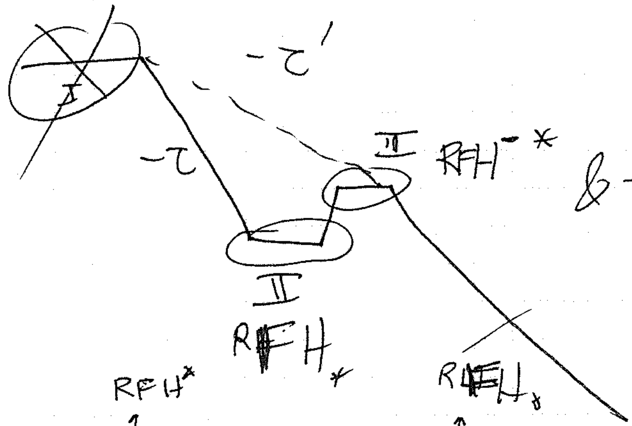
~~and this fits in~~ (cont'd)

and, this fits into an LES

$$\begin{array}{ccccccc}
\cdots SH_*(V,\partial V) & \longrightarrow & SH_*(V) & \longrightarrow & RFH_*(\partial V) & \to & \cdots \\
\text{P.D.} \wr\!\wr & & \text{P.D.} \wr\!\wr & & \text{P.D.} \wr\!\wr & & \\
\cdots SH^{-*}(V) & \longrightarrow & SH^{-*}(V,\partial V) & \longrightarrow & RFH^{1-*}(\partial V) & \to & \cdots
\end{array}$$

Proof:

V-shaped Ham. 1)  swap. $H \longleftrightarrow -H$, & do sth. at ∞stretch. ☒

2) 

& total cplx. is acyclic

$$\Rightarrow \quad 0 \to \underset{\text{ideal}}{\overset{RFH^*}{C_{III}}} \to \underset{\text{acyclic}}{C_{II,III}} \to \underset{\text{product}}{\overset{RFH_*}{C_{II}}} \to 0 \qquad ☒$$

$\Rightarrow$ has μ

$= C(f),\ f : C_{II} \to C_{III}[1].$

P.D. and products?

$\Rightarrow$ In this situation, $\mu|_{C_{III}}$ trivial on homology.

& a choice of chain htpy from $id_{C(f)} \to 0$ induces a secondary product, μ'_{III} on C_{III} (w/ shifted degree), for which f_* becomes a ring isomorphism.

(it follows that $[\mu'_{III}]$ is determined uniquely)

To summarize $\quad RFH_*(\partial V) \xrightarrow[\text{P.D.}]{\cong} RFH^{1-*}(\partial V)$

pair of pants $\longleftrightarrow$ secondary product μ'_{III} ; pair of pants is trivial

It follows that μ'_{III} is unital/assoc./comm. (From the def'n, it's not so obvious).

this equality & swaps Goresky–Hingston & string product (which are both part of this "pair of pants").

③ Restate in loop homology

How to phrase $RFH_*(S^*Q)$? · (Rmk: note $* \in (-\infty, \infty)$).

Thm: $RFH_*(\partial V) \cong H_*(\mathrm{Cone}(\varepsilon))$.

$V = D^*Q$: ε looks like: (or rather, $[\varepsilon]$ looks like):

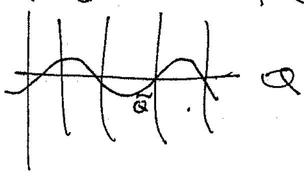
$$
\begin{array}{ccc}
H^{-*}(LQ) & \xrightarrow{\varepsilon_*} & H_*(LQ) \\
\downarrow c^* & & \uparrow c_* \\
H^{-*}(Q) & \longrightarrow & H_*(Q)
\end{array}
$$

where $c: Q \to LQ$ constant maps

(note this map can only live in degree zero.).

$$
\begin{array}{ccc}
H^0(Q) & \longrightarrow & H_0(Q) \\
\wr\wr & \nearrow & \\
H_n(Q) & \cap e(TQ) &
\end{array}
$$

e.g., count intersection of $Q \cap \tilde{Q}$



(note if $e(TQ)$ vanishes, this is just the zero map).

Can use this to define RFH_* in purely loop space terms.

In progress: should be able to describe a product on this complex, extending

λ & μ on $H^{-*}(LQ)$ & $H_*(LQ)$.

& the anomaly at constants disappears

& ~~if~~ e.g., if $e(TQ) = 0$, then GH product extends to $\cancel{H^\cdot(LQ, Q)}$ $H^\cdot(LQ)$?) (already know $H^*(LQ) / H^0(Q)$)

[Fukaya]: expect: $e(TQ)$ related to count of  constant annuli.