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[w/ A. Daemi].

(X,ω) cpct. sympl. m'fld.

$\mathcal{D} \subset X$ assume X kähler near D.

↑ smooth divisor (maybe normal crossings),

<u>Thm</u>: $\exists$ Filtered A_∞ cat. w/ obj. $L \subset X \setminus \mathcal{D}$ (<u>cpct.</u>) Lag'n rel. spin

using only hol. disks in $X \setminus \mathcal{D}$.

<u>Cor</u>: $\beta \in H_2(X \setminus D, L)$ L rel. spin

$\mu(\beta) = c\beta \cap [\omega]$ monotone $\mu \geq 4$ for β rep. by hol. disk.

$$\overset{\circ}{\mathcal{M}}_{k+1}(\beta) \qquad \beta \in \pi_2(X \setminus D, L)$$

$$\text{||}$$

$$\left\{ ((D^2, \vec{z}), u) \;\middle|\; \begin{array}{l} u : (D^2, \partial) \longrightarrow (X \setminus \mathcal{D}, L) \\ \vec{z} = (z_0, \ldots, z_k) \in (\partial D^2)^k \\ \text{disj. + respect cyclic order} \end{array} \right\}$$

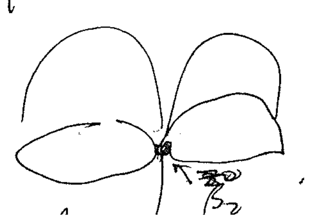
Main <u>Thm</u>: $\exists$ a compactification $\mathcal{M}^{RGW}_{k+1}(\beta)$,

① cpct, Hausdorff, metrizable

② kuranishi structure.

fiber product outside codim. 2 (doesn't affect VFC, so ok),

$$③ \; \partial \mathcal{M}^{RGW}_{k+1}(\beta) = \bigcup_{k_1 + k_2 = k+1} \; \bigcup_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1, \beta_2 \in \pi_2(X \setminus \mathcal{D}, L)}} \mathcal{M}^{RGW}_{k_1+1}(\beta_1) \; {}_{ev_i}\hat{\times}_{ev_0} \; \mathcal{M}^{RGW}_{k_2+1}(\beta_2)$$



"RGW" = "Relative GW."

An element of $\mathcal{M}_k^{RGW}(-)$ is: what?

first $((\Sigma, \vec{z}), u) \in \mathcal{M}_{k+1}(\beta) \leftarrow$ stable map compactification;

D tree like discs w/ tree-like spheres attached

$\uparrow$ -2 sphere in D.

Have forgetful map

$$\mathcal{M}^{RGW}_{k+1}(\beta) \longrightarrow \mathcal{M}_{k+1}(\beta) \quad \underline{\text{not surjective}},$$

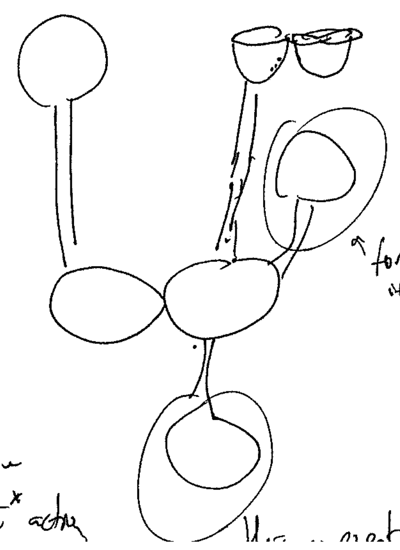
$\uparrow$ here

$((\Sigma, \vec{z}, u), \ell, m, (\sigma_\alpha))$

$\uparrow$ level fxn. $\uparrow$ multiplicity

(for normal crossings, have n level fns, n mults. & ...)

$X \setminus \mathcal{D}$ = compact $\cup$ $SN_{\mathcal{D}} X \times (0, \infty)$



$l = 0$ $\quad \emptyset$ in $X \setminus \mathcal{D}$ far from $\mathcal{D}$

$l = 1$ $\quad$ nhood of $\mathcal{D}$

$l = 2$ $\quad$ nhood of $\mathcal{D}$

$l = 3$ $\quad \vdots$

forbidden if concave

allowed b/c have $\mathbb{R}$ action in SFT + extra S^1 action $\rightsquigarrow \mathbb{C}^*$ action what's ℓ?

$$\Sigma = \bigcup_{\alpha \in A} \Sigma_\alpha \quad \text{where } \Sigma_\alpha = D^2 \text{ or } S^2.$$

↑ set of conn. component

level is a function

$$\ell : A \longrightarrow \mathbb{Z}_{\geq 0} \text{ satisfying:}$$

- $\ell(\alpha) = 0 \iff u(\Sigma_\alpha) \not\subset \mathcal{D}$.

(technical) • $\operatorname{Im} \ell = \{0, 1, \dots, |\ell|\}$ means $\nexists$ a gap (so if max im is $|\ell|$, also hits any $i < |\ell|$).

Define

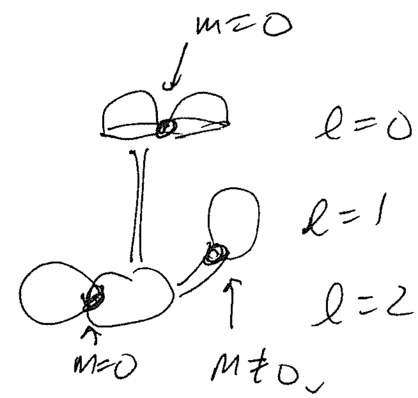
$DP :=$ double points of Σ

$p \in DP$ · $p \in \Sigma_\alpha \cap \Sigma_{\alpha'}$ (genus 0 so no self-intersection!)

Then multiplicity

① $m : DP \longrightarrow \mathbb{Z}_{\geq 0}$

② $\ell(\alpha) \neq \ell(\alpha') \iff m(p) \neq 0$. so:



③ Suppose $\ell(\alpha) = 0$ & $\ell(\alpha') > 0$.

$u_\alpha := u|_{\Sigma_\alpha} : \Sigma_\alpha \to X$ w/ image not in $\mathcal{D}$.

& $u_\alpha(p) \in \mathcal{D}$.

Then $m(p) :=$ int. multiplicity of u_α with $\mathcal{D}$ at p.

If both levels are > 0, m is extra information,

(3) Suppose Σ_α s.t. $l(\alpha) > 0$

(so $\Sigma_\alpha = S^2$).

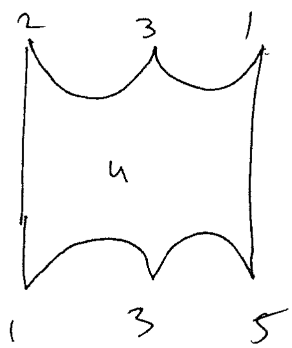
Define $W_+ = \{ p \in \Sigma_\alpha \cap \Sigma_{\alpha'} \mid l(\alpha') > l(\alpha) \}$

(& should match).

& $W_- = \{ \quad " \quad \mid l(\alpha') < l(\alpha) \}$.

Then, $$\sum_{p \in W_+} m(p) = \sum_{p \in W_-} m(p) + u([\Sigma_\alpha]) \cdot D$$ [balancing condition].

ex:



$\Rightarrow 9 = \deg \Sigma_\alpha + 6.$

This implies the following:

on $u_\alpha^* N_D X \longrightarrow S^2$, with $\mathcal{L} = u_\alpha^* N_D X$ (except $\mathbb{C}^*$ action). (degree of this $\mathcal{L}$ is $u(\Sigma_\alpha) \cdot D$).

$\exists\, s_\alpha$, a meromorphic section of this line bundle w/ zeroes on W_+ of mult. $m(p)$ & on W_- has poles of multiplicity $m(p)$, with no other poles or zeros

Last bit of data: $\left\{ (s_\alpha) \;\middle|\; \begin{matrix} \alpha \in A \\ l(\alpha) > 0 \end{matrix} \right\} / \sim$ sections as above

where $(s_\alpha) \sim (s_\alpha') \Longleftrightarrow \exists\, \rho_i \quad i = 1, \dots, |l| \quad \rho_i \in \mathbb{C}^*$

w/ $s_\alpha = \rho_{l(\alpha)} s_\alpha'$. one ρ_i per level.

(so simultaneous complex $\mathbb{C}^*$ change in each level).

when ρ_i goes to zero, create different level.

Here, codim is 2 # levels.

whereas, in stable map compactification, codim is $2\times$ # singular points.

<u>Topology</u>: say

$$((D^2_i, \vec{z}_i), u_i) \xrightarrow[i\to\infty]{} (((\Sigma_\infty, \vec{z}_\infty), u_\alpha), \ell_i, m_i, (s_\alpha)).$$

Requirements:

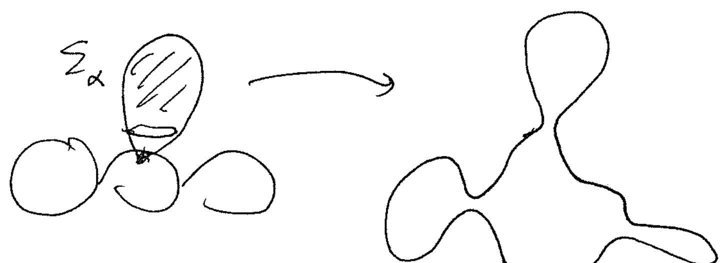
① $((D^2_i, \vec{z}_i), u_i) \longrightarrow ((\Sigma_\infty, \vec{z}_\infty), u_\infty)$ as underlying stable maps.

means ⓐ $\exists\ \vec{w}_i$ s.t. $(D^2, \vec{z}_i \cup \vec{w}_i) \rightsquigarrow (\Sigma_\infty, \vec{z}_\infty, \vec{w}_\infty)$

(to stabilize) as a stable curve

$$\underset{\text{component of } \Sigma_\infty}{\Sigma_\alpha} \setminus \varepsilon\text{-nhood of DP} = \Sigma_\alpha^\varepsilon.$$

then $\Sigma_\alpha^\varepsilon \xhookrightarrow[\Phi_{\alpha,i}]{} D^2$,



w/ $u_i \circ \Phi_{\alpha,i} \sim u_\infty$ ε-close in C^∞ sense.

as maps on $\Sigma_\alpha^\varepsilon$.

another condition: neck regions ε get very small.

② Further, $\exists\ \rho_{\ell,i} \in \mathbb{C}^*$ a sequence s.t.

Ⓘ $\rho_{\ell(\alpha),i} \circ u_i \circ \Phi_{\alpha,i} \xrightarrow{\text{converges}} s_\alpha$ on Σ_α. $\quad \rho_{\ell(\alpha),i} \circ u_i \circ \Phi_{\alpha,i}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Downarrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad s_\alpha.$

means: $s_\alpha \in \Gamma(u_\alpha^* N_D X)$

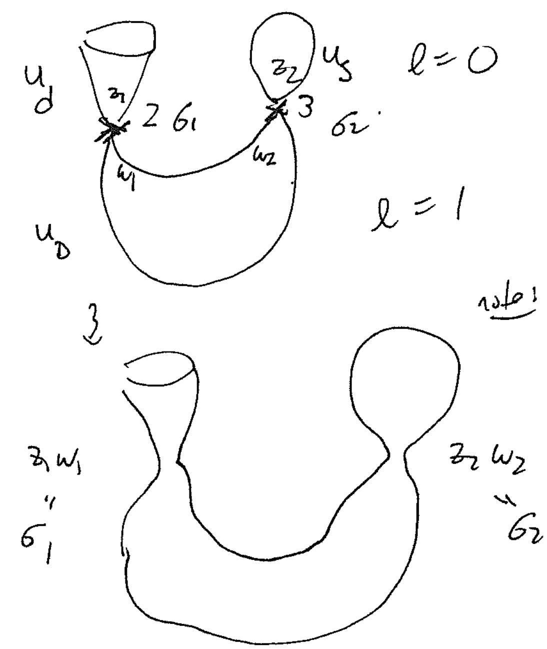
$\Downarrow$

(II) For $j > j'$

$$\lim_{i \to \infty} \frac{\rho_{j,i}}{\rho_{j',i}} \longrightarrow \infty .$$

Kuranishi structures : much same as before, but some subtleties in gluing analysis :

Ex: (*)



$\ell = 0$

$\ell = 1$

σ_1, σ_2 two defining parameters.

note: only two levels, so σ_1, σ_2 cannot be completely independent, b/c $\ell = 0, 1 \Rightarrow$ codim 2.

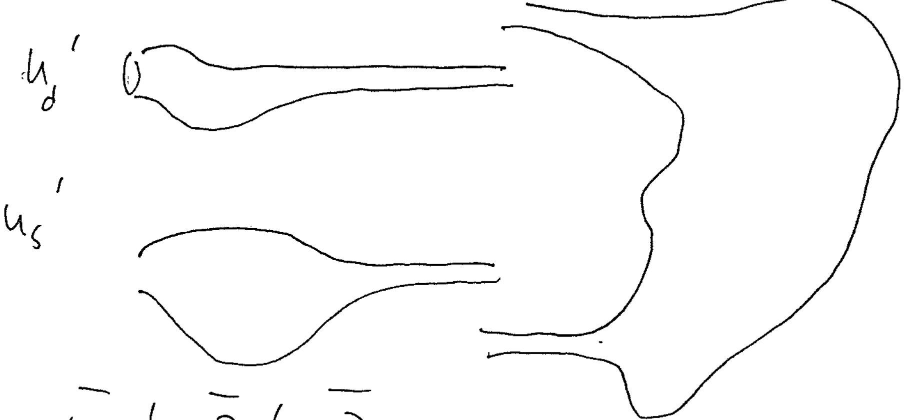
ok: b/c note u_D has 1-d freedom.

condition: "$\sigma_1^2 = \sigma_2^3$."

this is a bad condition: b/c suggests your neighborhood of (*) is a cusp!

So, need an extra type of Kuranishi chart to handle it:

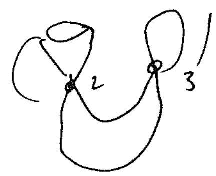
Inconsistent solution:



w/ $\bar\partial u_d' = \bar\partial u_S' = \bar\partial u_D' = 0$.

Results:

(a) Moduli of inconsistent solutions is a smooth mfld $\mathcal{M}_{inconst}$.

if original configuration  is Fredholm regular.

(or obstruction bdle, etc.

(b) $\exists$ a map

$$\mathcal{M}_{incont} \longrightarrow \mathbb{C}$$ smooth map. (or $\exists$ a kuranishi map, etc.)

$$\rho_1 - \rho_2 .$$

zero set: $\rho_1 = \rho_2$ is elt. of actual moduli spaces, one wants to study.

Explanation of (a) sketch: uses "alternating method" of gluing :