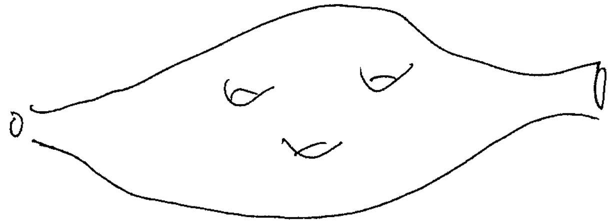
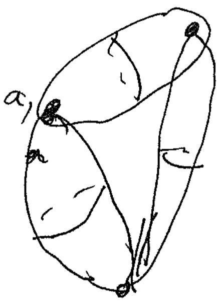
Y. Lekili, Mirror symmetry for punctured surfaces (joint w/ A. Polishchuk).

A-side



$\Sigma_{g,d}$ ← # punctures; g ← genus

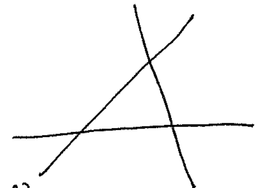
B side Nodal stacky curve.



$b = k_3,\ l_3$ $\mathbb{P}^1(a,b)$ ∪

$k_1 = a,\ + l_1$ $\mathbb{P}^1(b,c)$ ∪

$c = k_2 + l_2$ $\mathbb{P}^1(c,a)$.



Local model near nodal points (k, ℓ):

$\{xy = 0\} / \mathbb{Z}_k$. where, if

ξ denotes a kth root of unity, then

$\mathbb{Z}_k$ acts by $(x,y) \longmapsto (\xi x, \xi^{\ell} y)$.

$[\ell = 1]$: Sibilla–Treumann–Zaslow

($\ell = 1$ always $\Rightarrow$ only get genus 0, 1 surfaces).

Call such a curve $C_{(k,\ell)}$. either a cycle  or linear .

Main Result: $\mathcal{W}(\Sigma_{g,d}) \simeq D^b Coh(C_{(k,\ell)})$. (wrapped)

$\cup$ $\cup$

$\mathcal{F}(\Sigma_{g,d}) \simeq Perf(C_{(k,\ell)})$. (but this follows from first statement w/ more work)!

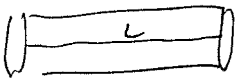
cpct. Fukaya category

Rmk: There's a fn. $\{(k,\ell)'s\} \xrightarrow{f} \{(g,d)\}'s$ surjective on $d \neq 0$.

that goes as above. But, f is not injective,.

$\exists$ different (k,ℓ)'s that give same (g,d)'s

Ex:

 $\longleftrightarrow$ $\mathbb{C}^*$

T^*S^1

$Hom(L,L) = \mathbb{C}[z, z^{-1}]$,

but, need to choose a grading;

e.g., choose $\Omega = \frac{dz}{z}$ w/

$|z| = 0$.

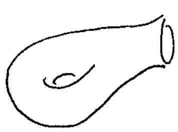
If choose different Ω, mirror is not $\mathbb{C}^*$. (in fact $\nexists$ S^1's in $T^*S^1, \tilde{\Omega}$; mirror has no points)

there are $H^1(\Sigma_{g,d})$ many grading choices, a priori.

& point is, ~~like~~ $\frac{dz}{z}$, ~~these~~ can be thought of as a canonical choice.

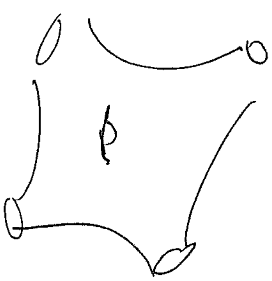
(need to pick some specific choices on $\Sigma_{g,d}$ to make this hold, or work over $\mathbb{Z}/2$)

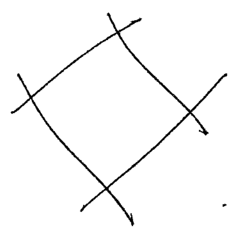
First case:

 [L-Perutz] w/ flat metric extending to T^2 ($\Rightarrow \Omega$),

$\xleftrightarrow{HMS}$  $y^2 + xy = x^3$

— curves mapped + cpct.

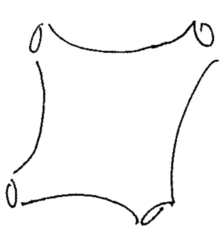
 [L-Polishchuk] + [heating]



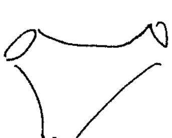
which should be birationally equivalent?

also not unique: many choices w/ different 3-folds, W

$g=0$ [Abouzaid-Auroux-Efimov + Katzarkov-Orlov]: $\mathcal{W}(\Sigma_{0,d}) \simeq MF(\text{toric 3-fold}, W)$.

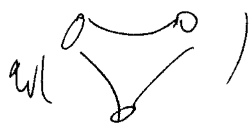


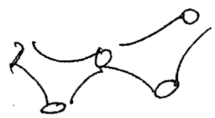
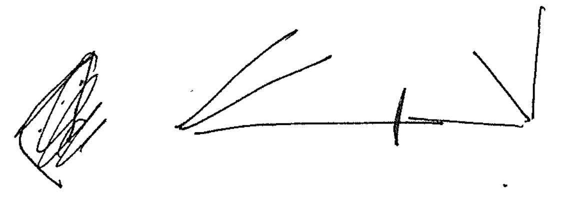
$\mathbb{Z}/2$-graded. (+ pick $\mathbb{C}^*$ action on RHS to fix gradings)

ex:  $\longleftrightarrow$ $(\mathbb{C}^3, xyz)$ & grading from $\mathbb{C}^*$ action.

[these all use ...ical 'deformation-theoretic'

[Sheaf-theoretic approach :] [Sibilla-Treumann-Zaslow
Heather Lee
Pascaleff-Sibilla
Nadler].

Idea: prove $\mathcal{W}($  $) \simeq MF(\mathbb{C}^3, xyz)$, 

& build  by gluing: 

The result here uses another approach:

[Resolution of singularities]:

Say C algebraic variety, proper (projective),

$\rightsquigarrow Perf(C)$ knows a lot about C, e.g., whether C is smooth.
($\Leftrightarrow Perf(C)$ is "smooth").

If C not smooth, $Perf(C) \subsetneq D^b Coh(C)$ proper subcat.
$\uparrow$ always a smooth category [Lunts]
yet, it's not proper.

So,
$$\underset{\text{not smooth}}{Perf(C)} \hookrightarrow D^b Coh(C).$$

On the mirror side,
$$\underset{\text{almost never smooth}}{\mathcal{F}(X)} \longrightarrow \mathcal{W}(X) \quad \text{always smooth}$$

Smoothness helps compute ext groups (only need finite resolutions)

To deal w/ non-properness, instead consider

$$\mathrm{Perf}(C) \underset{\text{faithful}}{\overset{\text{full}}{\hookrightarrow}} D^b(\text{mod-}A) \xrightarrow{\text{localizes}} D^b(\mathrm{Coh}(C)).$$

↑ smooth proper (depends on choices)

nc categorical resolution [van den Bergh, others.]

On the A-side:

$$\mathcal{F}(X) \underset{\text{faithful}}{\overset{\text{full}}{\hookrightarrow}} \mathcal{W}(X, \Delta) \xrightarrow{\text{localization}} \mathcal{W}(X).$$

↑ partially wrapped category [Auroux, LOT, Sylvan, GPS] Seidel's FS categories, orlov, landau ...

(smooth + proper if choose Δ carefully)

Specializing to $\Sigma_{g,d} / C_{(n,e)}$, use:

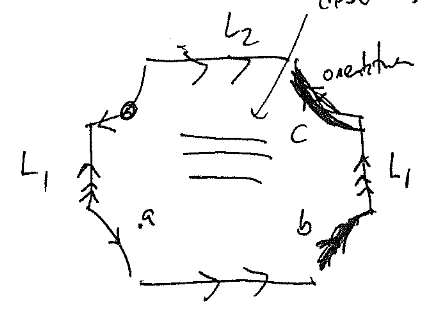
- Burban–Drozd on the B-side
- Haiden–Kontsevich–Katzarkov on the A-side

to study middle category.

(middle structure has finitely many objects, & no A_∞ structure] :

* Bocklandt (predecessor of [HKK])

ex:



determines grading

orientation

$\mathcal{W}(1;1)$ genus, 1 bdry w/ 1 marked point

is not an FS category.

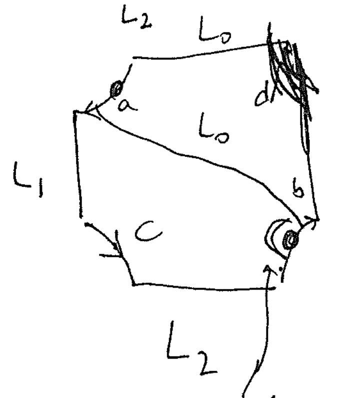
a, c, b are morphisms, so get:

$cb = 0$

$ba = 0$

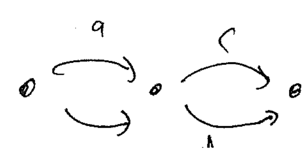
$|a| = |c| = 0, \ |b| = 1.$

or:



Cover surface by discs s.t. each disc has <u>one</u> marked point on boundary

note that this object supported



$cd = 0 = db$

$\mathcal{F}(\Sigma_{g,d}) \simeq \mathrm{Fun}^{ex}(\mathcal{W}(\Sigma_{g,d}), \mathrm{Perf}\,\mathbb{k})$.

What is $\mathcal{A}$?

Auslander order:

A algebra $\qquad$ M A-module

$$\mathrm{End}_A(M) = \Lambda$$

& there's an obvious map

$$\mathrm{mod}\ A \xrightarrow{f} \mathrm{mod}\text{-}\Lambda$$
$$N \longmapsto \mathrm{Hom}_A(M, N).$$

want: - Λ has finite global dimension (2 in this case).

- f to be full & faithful. ($\Rightarrow$ [Auslander] gives smooth proper resolution).

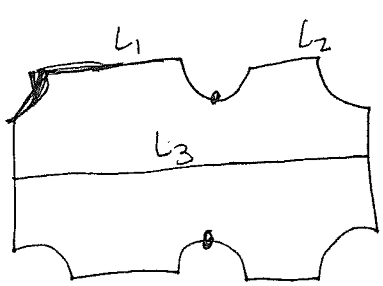
If A has finitely many indecomposables, take $M = \bigoplus$ indecomps.

If A has finitely many indecomp. Cohen-Macaulay modules, still ok.
(depth = dimension).

Care specifically about

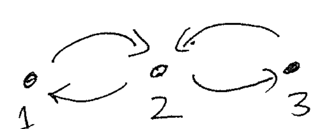
$A = k[x,y]/xy=0$, & the sum of the three indecomp. Cohen-Mac modules are: $M = A \oplus \mathbb{k}[x] \oplus \mathbb{k}[y]$.

Then, $\Lambda = \mathrm{End}_A(M)$ $\rightsquigarrow$ mirror



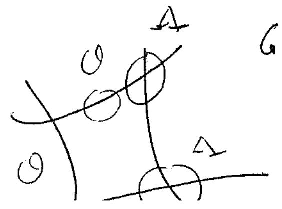
& $\mathrm{End}(L_1 \oplus L_2 \oplus L_3) = \Lambda$.

compute:



& 12's can't be composed w/ 23's.

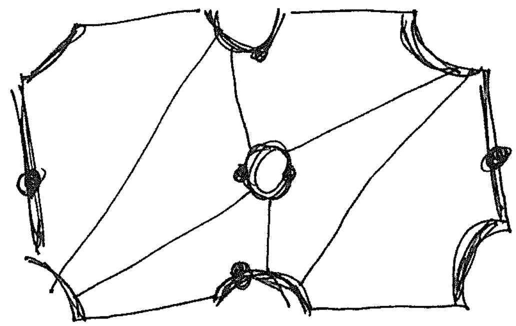
Auslander order: $\mathcal{A}$: sheaf of algebras on



~~[Auslander]~~

Globally, $\mathcal{A} = \mathrm{End}_{\mathcal{O}}(\mathcal{O} \oplus \mathcal{I})$
ideal sheaf of

key example:



:= boundary :

so, have: (4-punctured sphere)

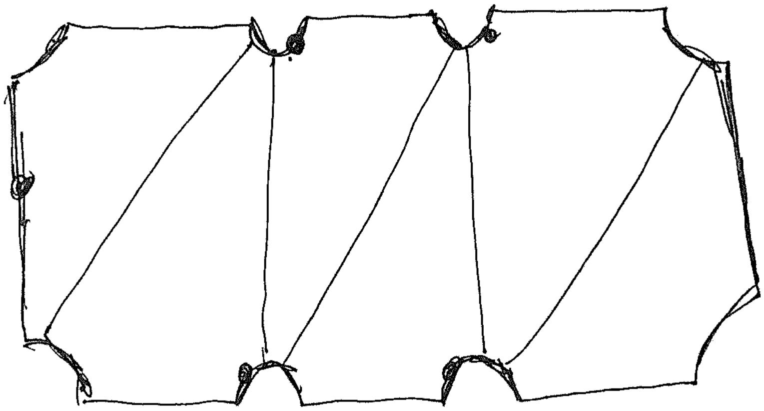
$\longleftrightarrow$

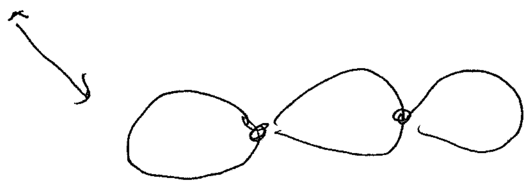
$\mathbb{Z}_2$ stacky point b/c 

Another 4-punctured sphere: braiding the gluing $\longleftrightarrow$ $(k, \frac{l}{r})$.

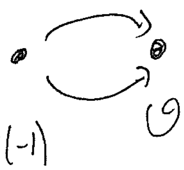
combinatorially the same, & homeo between surfaces preserves line fields

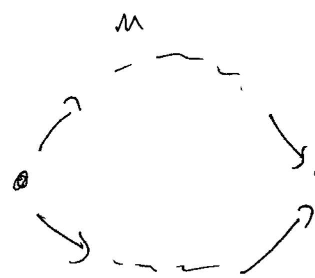
$\Rightarrow$ categories are the same.





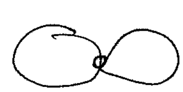
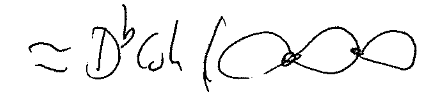
$\mathbb{P}^1$:





$\mathbb{P}^1(m,n)$

Hard pick objects so algebras match on the nose.

Localizing away from points; get $D^b Coh$ () $\simeq D^b Coh$ ()

$\mathbb{Z}/2$ not obvious

Local picture for

$\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$

[Sibilla]:

$\mathbb{C}^2/\mathbb{Z}_2$ $\xleftarrow{\text{blow-up}}$

$\swarrow xy$

$[\mathbb{C}^2/\mathbb{Z}_2]$

[Derived McKay] $\Rightarrow$ equivariant sheaves by defn.

$D^b Coh(\widetilde{\mathbb{C}^2/\mathbb{Z}_2}) \simeq D^b Coh([\mathbb{C}^2/\mathbb{Z}_2])$