E. Murphy, Pruned Arboreal Singularities & loose Legendrians.

Thm: Let $L \subseteq \mathbb{C}^n$ be a pruned arboreal singularity, and $\Lambda \subseteq S^{2n-1}$ its link.
Then, Λ is loose iff every constructible sheaf w/ SS. on L is constant.

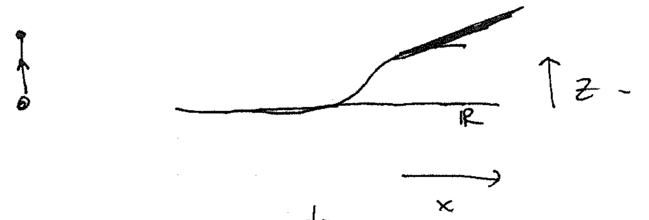
Arboreal singularities: (Nadler). Fix dimension n.

Start with a rooted tree T. w/ # vertices < $\underset{n+1}{\text{dim}+1}$

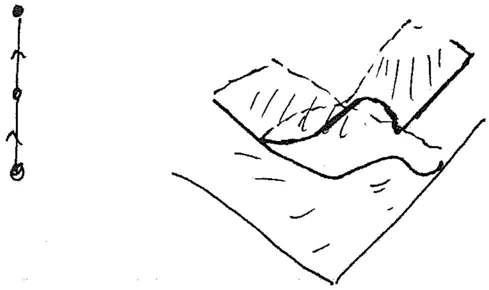
- root $\rightsquigarrow \mathbb{R}^n \subseteq \mathbb{C}^n$
- each additional vertex $\rightsquigarrow \mathbb{R}^n_+$ attached to all vertices below it. (w.r.t. adjacency)

(then multiply to stabilize by $\mathbb{R}^k \subseteq \mathbb{C}^k$.

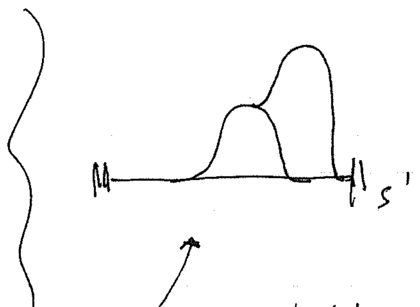
Ex: (using front diagrams)



$y = \frac{dz}{dx}$.



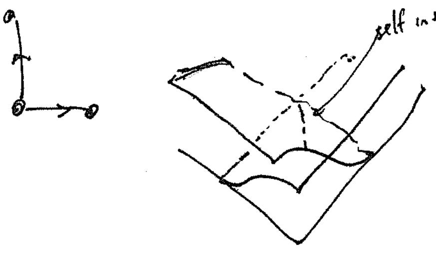
plane
(sing surface in $\mathbb{R}^3$)



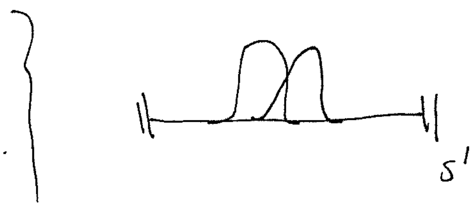
(adding a root $\longleftrightarrow$ conormal cone + zero section)

in $J^1S^1 \subset S^3$

link: (intersect w/ cylinder at ∞)
(front of link)

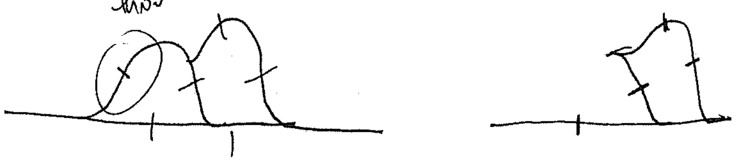


self intersection in front, but immaterial (b/c transverse isection, so don't intersect when lifted.



Pruned arboreal: An arboreal singularity has a Whitney stratification. Throw away any collection of connected components of top dim'l open stratum.

ex:



(& maybe throw away lower-d. strata when they're not connected to anything.

Motivation:

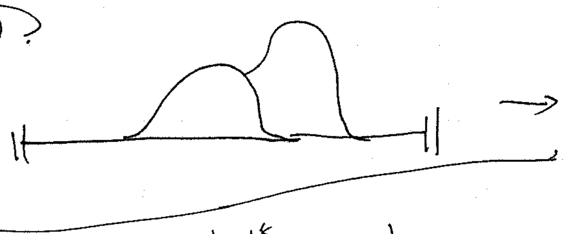
- Any Lag'n singularity can be deformed to arboreal (Nadler)
- Any Weinstein has an arboreal skeleton (Eliashberg-Nadler-Starkston)

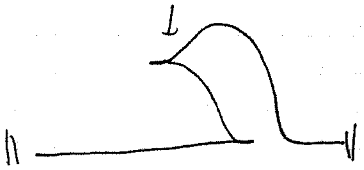
(but these results crucially require more general pruned arboreal singularities.)

But creates free ∂?

parabolic tangencies, not semi-cubic (in fact not smooth)

"loose"?

ex: 



(homeo. to S^1, so $\partial L \underset{homeo}{\sim} D^2$, but not smooth.)

Rmk: this is a singular cusp, but it's as good as a smooth cusp for the purpose of loose Legendrians / h-principle.

(don't matter, b/c C^1-closeness of fronts / C^0 closeness of lift is all that's relevant for h-principles).

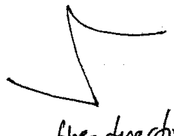
Context: Flexible Weinstein manifolds.

Thm (Cieliebak-Eliashberg): If X_1, X_2 are flexible Weinstein manifolds, ~~...~~ & $X_1 \underset{diffeo.}{\cong} X_2$ inducing an iso. $TX_1 \underset{\mathbb{C}}{\cong} TX_2$ (as almost cplx.),

then $X_1 \underset{sympl.}{\cong} X_2$.
(Weinstein htpic)

If X is a Weinstein manifold, and X has a pruned arboreal skeleton, and each component of top. dim'l stratum is adjacent to a singularity w/ loose link, then

$\rightsquigarrow$ X is flexible.

Loose: each component of top strata has  is front.

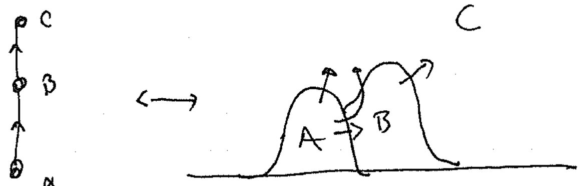
($\Rightarrow \infty$ loose nbhd in cofiber direction).

Rmk: everything today in dim > 4,

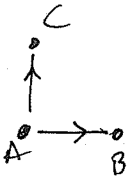
False Conj: If $\mathcal{W}(X) \cong 0$ then X is flexible.

(but it seems like the local version of this conjecture is true!)

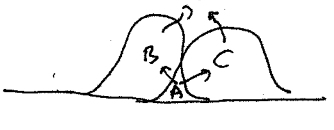
Thm (Nadler): Sheaves on arboreal singularity associated to T are $\cong$ representations of T as a quiver.



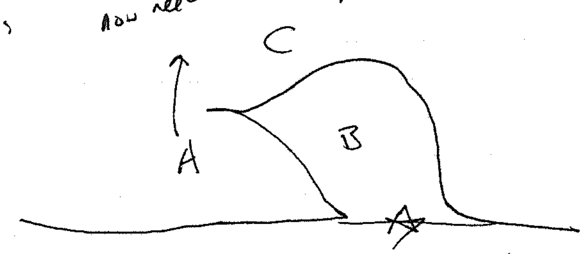
(sheaf on $\mathbb{R}^2$ w/ ss. on Leg.).

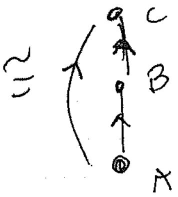


D determined by $A \to B$, $A \to C$.



pruning: now needs to be a quasi-iso.

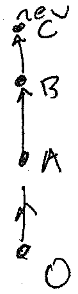




so reduces space of rep'ns

Rmk: either not a-stabs, or force to be zero here

or: actually: choose a new initial object in quiver:



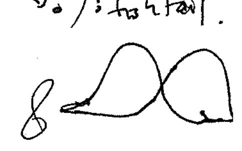
expands # localizations

for instance, deleting ☆ forces $O \xrightarrow{\cong} B$ . .

Pf: Alg: characterize which localizations cause the quiver to have no rep'ns.

geom: check this corresponds to loose things.

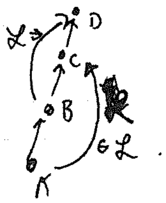
(appears in deleting f....): fishtail.



Let J be a tree, and let $\mathcal{L}$ be a subset of morphisms. (Assume J linear). &

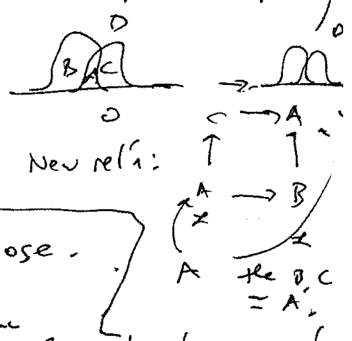
Let $(\mathrm{Ob}\, J)/\mathcal{L}$: (a) if $A \xrightarrow[\in \mathcal{L}]{} B$, say $A \sim B$.

(b) if



, then $A \sim B \sim C \sim D$. .

nonlinear case: need one more relation. (b/c not localizing quiver, but rather a pushout completion)



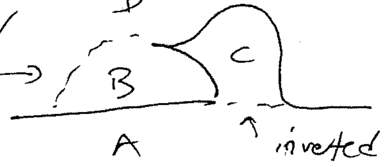
Prop (linear case): (nonlinear case is a little more complicated).

① If $J[L^{-1}]$ has no rep'ns then $\mathrm{Ob}(J)/\mathcal{L} = 1$ point. .

② If $\mathrm{Ob}(J)/\mathcal{L} = 1$ pt, then the corresponding Λ is loose.

To prove ①: if Ob has > 1 point, find a sequence w/ nothing ~~[illegible]~~ isomorphic, & construct a rep'n.

to prove ②: If (b) exists, have a contact slice w/



"quotienting quivers" ⟷ "finding contact slices"?