T. Perutz, Floer theory in spaces of stable pairs over Riemann surfaces

joint w/ Andrew Lee

I, Background + Goals

Z Riemann surface (cpct, connected) $\qquad \Lambda \to Z$ hol. line bdle, deg. $d \geq 0$.

$N_\Lambda = \{$ rk. 2 semi-stable hol. bdles $E \to Z$ with $\det \cong \Lambda \}$.

$\forall$ line bdles $F \subset E$, $\deg F \leq \frac{d}{2}$.

(stable: $<$).

Proj. alg. variety of dimension $3g-3$ $(g>1)$, non-singular on stable locus,

but typically singular when $=$ occurs.

So: when d is odd $\Rightarrow$ smooth

d is even $\Rightarrow$ not smooth.

Thm [Narasimhan–Seshadri]: $N_\Lambda \cong \{$proj. flat $U(2)$-connections$\}/$gauge equivalence.

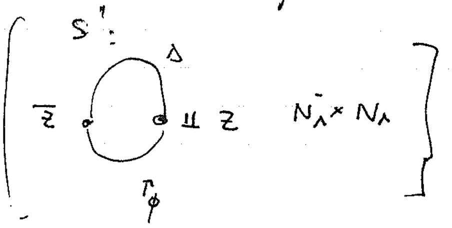
Atiyah–Floer idea:

$Y^3 \supset Z$, instanton Floer hom. $I_*(Y)$ interpreted as sympl. Floer homology of some kind in $N_\Lambda(Z)$.

Fibered case: $Z \hookrightarrow Y \to S^1$

$$I_*(Y;\ U(2)\text{-bdle of odd degree } d \text{ over } Z) \underset{\text{Dostoglou–Salamon}}{\cong} HF_\bullet(N_\Lambda, \text{action of monodromy on } N_\Lambda)$$

fixed pt. Floer homology (symplecto.)



d even: neither side makes sense thanks to singularities.

Heegaard splittings: $Y = U_0 \cup_Z U_1$.

$d=0$:

$I_*(Y) \cong$?? Lagn Floer hom. in N_Λ.

say a $\mathbb{Z}HS^3$

Workarounds:

- equivariant formulations ... ?
- Don't work mod gauge [Salamon–Wehrheim].
- Surgical adjustments to Y [Kronheimer ...] (e.g., $\# T^3$ + a bdle on it).

• Replace N_Λ,
e.g., extended moduli space [Manolescu-Woodward, Daemi-Fukaya].

Today: another proposed replacement for N_Λ:

M_Λ – a certain space of semistable pairs in sense of [Bradlow, Thaddeus].

$$\begin{array}{c} M_\Lambda \\ \downarrow \\ N_\Lambda \end{array}$$

smooth, projective, $\dim_{\mathbb{C}} 3g$, Fano (so monotone).

$$M_\Lambda = \left\{ (E,\phi) \;\middle|\; \begin{array}{l} E \text{ rank 2 } \overset{\text{semistable}}{\text{s.stable}} \text{ bdle, } \det \Lambda \\ \phi \in H^0(E)\setminus 0 \\ \text{If } F \subset E \;\&\; \phi \in H^0(F) \\ \text{then } \deg F \neq \frac{d}{2} \end{array} \right\} \Big/ \text{iso.}$$

but, now M_Λ is smooth.

AJ (forget ϕ) $\downarrow$

N_Λ

Take $\deg \Lambda = d = 2g+2$.

In that case, $\dim_{\mathbb{C}} M_\Lambda = 3g$, & generic fiber of AJ is $\mathbb{P}^3$.

Smooth, projective, Fano. (proved rank 2).

[Bradlow + Bradlow-Daskalopoulos]:

$$M_\Lambda \cong \left\{ \begin{array}{c}\text{rank 2 vortices} \\ \text{over } Z\end{array} \right\} \Big/ \text{gauge}.$$

Just as proj. flat connections on Z is a dim'l reduction of instanton eq'n in 4d
• flat connection in 3d,

the vortex equation is a reduction of
• $u(2)$ SW eq'ns in 4 or 3d [Feehan-Leness].

II. The stable pairs spaces [Thaddeus]:

Fix $\sigma \in \mathbb{R}$, $\sigma > 0$. Then, define $M_{\Lambda,\sigma} = \left\{ (E,\phi) \;\middle|\; \begin{array}{l} E \text{ rk. 2 hol. vec. bdle,} \\ \phi \in H^0(E)\setminus 0 \end{array} \right\}$

s.t.: $\forall$ lines $F \subset E$, ① $\deg F \leq \frac{d}{2} + \sigma$

② $\deg F \leq \frac{d}{2} - \sigma$ if $\phi \in H^0(F)$.

d even (say, for notational convenience to avoid $L - \rfloor$ symbols):

$(E,\phi) \in M_{\Lambda,\sigma}$ with $\sigma \notin \mathbb{Z}$.

• $\phi \in H^0(F_\phi)$ some $F_\phi \subset E$, so (by some Lemma) $\deg F_\phi \geq 0$ & ② $\Rightarrow \sigma \leq \frac{d}{2}$ if $M_{\Lambda,\sigma} \neq \emptyset$.

"First" moduli space: $\sigma = \frac{d}{2} - (\text{small})$

$\textcircled{2} \Rightarrow \deg F_\phi = 0$ so $(F_\phi, \phi) \cong (\mathcal{O}, 1)$.

So, $0 \to \underset{\substack{\Psi \\ 1}}{\mathcal{O}} \longrightarrow E \longrightarrow \Lambda \longrightarrow 0$

non-split extension by $\textcircled{1}$.

So, $M_0 \cong \mathbb{P} H^1(\Lambda^{-1}) \cong \mathbb{P} H^0(K\Lambda)^*$ (serre duality)

$\hookleftarrow Z$ via $|K\Lambda|$

Next: M_1 $\quad \sigma = \frac{d}{2} - 1 - (\text{small})$

$\deg F_\phi \leq 1$

(if 0, as before, if 1, have curves worth of $(\phi$'s$)$):

conclusion: $\begin{matrix} M_1 = Bl_Z(M_0). \\ \downarrow \\ M_0 \end{matrix}$

& $\underline{M_i} := \frac{d}{2} - i - (\text{small})$

$i > 1$, M_{i+1} differs from M_i by a *flip*.

$$\begin{tikzcd}[column sep=small] & \circ \arrow[dl] \arrow[dr] & & \circ \arrow[dl] \arrow[dr] & & & \\ M_1 \arrow[d] \arrow[dr] & & M_2 \arrow[dl] & & M_3 \arrow[r, dashed] & M_{top} \arrow[d] & \sigma \in (0,1). \\ M_0 & M_{crit} & & & & N_\Lambda & \end{tikzcd}$$

(M_{top}: E semi-stable & F_ϕ does not destabilize it).

- smooth projective of dim. $d+g-2$.
- $i > 0$, $\mathrm{Pic} = H^2 = \mathbb{Z}^2$.

For $d > 2g-2$, AJ is surjective "resolution"

III Our choice

$M_Z \begin{cases} \bullet\ M_{top} - \text{"close" to world of flat connections, (b/c vortex eq'ns small perturb of flat conn. eq'ns?)} \\ \bullet\ d = 2g+2 \end{cases}$

(means: to get a cobordism for S^2 to Z so $M_\Lambda(S^2) = pt.$)

(b/c: right degree for $u(2)$ SW eq'ns on punctured handle bodies to define (immersed) Lag'ns in $M_{\Lambda,\sigma}$ $\quad U \quad \partial U = Z$.

[cf., Heegaard Floer lines in $Sym^g Z$ (right dimension for SW eq'ns on punctured handlebodies to define Lag'ns in $Sym^g Z$).]

Good omen: $M_{\Lambda,\sigma}$ is Fano specifically for M_{top}, $d \approx 2g+2$ ($2g+1$ ok). desired vortices?

Conj: A handlebody U, $\partial U > Z$ determines an embedded Lag'n $L_U \subset M_Z$. True for $g \leq 1$.

$(S^3)^g$ $/\mathbb{Z}_2$ various for $g=0$. (true for $g=1$ [Smith]).

[Basis for Heegaard-Floer construction].

IV Test case:

Fibered 3-manifolds, sympl fixed points.

$\Gamma = \pi_0 \, \mathrm{Diff}^+(Z)$

$$\tilde{\Gamma} = \pi_0 \left\{ \begin{array}{l} \hat{\phi} = (\phi, \tilde{\phi}), \text{ where} \\ \phi \in \mathrm{Diff}^+ Z \\ \tilde{\phi} : \Lambda \xrightarrow[\cong]{C^\infty} \phi^* \Lambda . \end{array} \right.$$

There's an obvious surjection

$$1 \to \underset{\substack{\uparrow \\ \text{linearizations of } 1}}{H^1(Z,\mathbb{Z})} \to \tilde{\Gamma} \to \Gamma \longrightarrow 1$$

& there's naturally $\tilde{\Gamma} \xrightarrow[\text{homom.}]{\Theta} \frac{\mathrm{Symp}}{\mathrm{Ham}} (M_{\Lambda,6}, \text{kähler form})$.

(by doing the construction $M_{\Lambda,6}$ in families & taking monodromy).

On M_Z, $\Theta(\hat{\phi})$ is a <u>monotone</u> symplectic automorphism.

& define $\boxed{HSP_*(\hat{\phi}) := HF_*(M_Z, \Theta(\hat{\phi}))}$

stable pair homology ... fixed point Floer

- finitely gen. $\mathbb{Z}$-mod, $\mathbb{Z}/2$-graded.
- $QH^*(M_Z)$-mod ($q = 1$ &, so another fin. gen. abelian group!)

$\underline{g=1}$: $M_Z := Bl_Z(\mathbb{P}^3)$ Z embedded via $|\Lambda|$, $\deg \Lambda = 4$.

Fano

<u>Results</u>: [Lee - P.]

$\underline{g=1}$ • $QH^*(M_Z) \underset{\text{ring}}{\cong} \underset{QH^*(\mathbb{P}^3)}{\mathbb{Z}[u]/(u^4=1)} \oplus \overbrace{H^*(Z)}^{\text{gen. eigenspace for } c_1 * -, \text{ eigenvalue } \lambda = -1}$

$\mathbb{Z}^4$ even. 1/2 odd. [c.f., Smith]

(in ordinary $H^*(M_Z)$, lose ring structure on $H^*(Z)$!)

• $HSP(\hat{\phi}) \cong \mathbb{Z}^4_{\text{even degree}} \oplus \underbrace{HF_*(Z, \phi)}$

as $QH^*(M_Z)$ modules. $c_1 * - = -1$ (gen. eigenspace

<u>Rmk</u>: • $HF(Z, \phi)$ is independent of the symplectic representative ϕ when $\phi^* - Id$ acts invertibly on $H^*(Z, \mathbb{R})$ (& most mapping classes satisfy this).

• If not (e.g., ϕ = power of Dehn twist) make a choice "power of symplectic Dehn twist".

- $HF(Z, \phi)$ explicit
- It's a certain summand in the monopole Floer homology of T_ϕ [+ monotone perturbation]

Pf: small blow-up compute by hand, & continuity argument to get to Fano point (~~contin~~ bifurcation cf. [Y. Lee]).

then use fact that a Dehn twist on Z turns into a sympl. Dehn twist on S^3 (in the $\lambda = -1$ summand), by Smith's argument).