Z. Sylvan, Spherical functors from Legendrian isotopy

Def: A Legendrian $\Lambda \subset Y$ is swappable if $\Lambda_+ \sim \Lambda_-$ in $Y \setminus \Lambda$. (Λ_+ = pos. Reeb pushoff, Λ_- = neg. Reeb pushoff)

A choice $\Lambda_+ \overset{\gamma}{\sim} \Lambda_-$ induces $\phi_\gamma : \Lambda \to \Lambda$ depends on γ.

Q: Can you find Λ swappable in two ways? (so that $\phi_\gamma, \phi_{\gamma'}$ are not homotopic?)

Rmk: If so, obstructs the fillability of Y.

Def: A stop $\sigma \supset \partial M$ (Liouville domain) is a hypersurface with boundary s.t. $(\sigma, \lambda|_\sigma)$ is a Liouville domain

Two main examples:

- $\Lambda \subset \partial M$ Legendrian $\rightsquigarrow$ σ_Λ (or Λ) is a stop
  $T^*\Lambda \hookrightarrow J^1\Lambda$, $\sigma_\Lambda \cong \mathrm{im}(T^*_{\epsilon}\Lambda)$

- $W : M \to \mathbb{C}$ superpotential, then $W^{-1}(\text{a pt at } \infty) \hookrightarrow \partial M$ is a stop.
  (under nice conditions on W, M).

Note: σ has a std. nhood
$U : \sigma \times \{\mathrm{Re}\, z \geq -1\} \hookrightarrow \hat{M}$ (completion), (σ sits at 0 say — extends to a proper embedding using $\mathrm{Re}\, z \geq 0$!)

& $\overline{M} \setminus U$ is a Liouville sector.

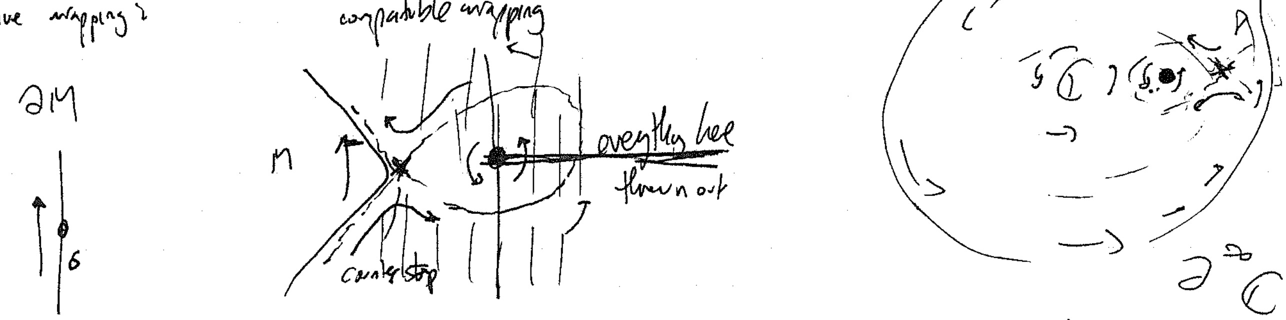
& given such a sector, can recover M, σ back.

So, we will call the pair (M, σ) a sector.

Def. The (partially) wrapped Fukaya category $\mathcal{W}(M,\sigma)$ (or $\mathcal{W}_\sigma(M)$) can be defined as a subcategory of $\mathcal{W}(M)$ by using a nice wrapping flow which

this is sufficient — [• preserves $\partial\sigma$ • is positively $\pitchfork$ to $\mathrm{int}(\sigma)$] and discard all objects & morphisms which meet σ.

Naive wrapping?



(projecting to $\mathbb{C}$, in σ coords, ordinary wrapping)

Sketch of thm [Ekholm-Lekili]:

There is a particular fully faithful embedding

$$LDGA\langle \Lambda ; C_\bullet(\Omega\Lambda)\rangle \hookrightarrow \mathcal{W}(M, \sigma_\Lambda)$$

↑ w/ non-central coeffs.

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2) F Liouville domain $\rightsquigarrow$ $\Sigma F = F \times \mathbb{D}$, $\sigma = F \times \{-1,1\}$.

$$\cong F \times T^*[0,1]$$

Induces a stabilization functor

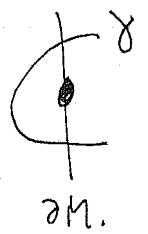
$$\Sigma : \mathcal{W}(F) \longrightarrow \mathcal{W}(\Sigma F, \sigma).$$
$$L \longmapsto L \times \gamma.$$

Fact: When F has enough Lagrangians, Σ is an equivalence.

Rmk: We'll assume σ's have enough Lagrangians always.

Def: $\sigma \subset \partial M$ induces $i_\sigma : \mathcal{W}(\sigma) \to \mathcal{W}(M, \sigma)$ "Orlov functor"

On objects, $L \subset \sigma \longmapsto$



Expectation:

$$\begin{tikzcd}
C_*(\Omega\Lambda) \arrow[rr, "\text{Abouzaid}"] \arrow[d, "\text{coeffs. id}"'] & \circlearrowright \text{ diagram commutes} & \mathcal{W}(T^*\Lambda) \arrow[d, "i_{\sigma_\Lambda}"] \\
LDGA\langle \Lambda, C_\bullet(\Omega\Lambda)\rangle \arrow[rr, "E-L"] & & \mathcal{W}(M, \sigma_\Lambda)
\end{tikzcd}$$

, Assume this for now!

3) Def/Thm [Anno-Logvinenko]: A functor $f : A \to B$ with left and right adjoints l, r satisfying two of the following conditions is spherical (in which case, it satisfies all conditions)

a) The twists t, t' for $f \circ r \xrightarrow{\text{counit}} \mathrm{Id}_B \to t$
$t' \to \mathrm{Id}_B \xrightarrow{\text{unit}} f \circ l$
are inverse equivalences.

b) The cotwists m, m': $m \to \mathrm{Id}_A \to r \circ f$
$l \circ f \to \mathrm{Id}_A \to m'$
are inverse equivalences.

c) $l \circ t \cong r[1]$ in a particular way.

d) $r \cong m \circ l[1]$ in a particular way. (& adjoints of sphericals are sphericals)

Thm [Abouzaid - Ganatra]: The structure functors of the Fukaya category of an LG model Seidel:

$$\begin{tikzcd} FS(W) \arrow[loop left, "{t,t'}"] \arrow[r, bend left, "\cap\ \text{[Seidel restriction]}"] & F(W^{-1}(pt.)) \arrow[l, bend left, "U = \text{Orlov}"] \arrow[loop right, "{m,m'}"] \end{tikzcd}$$

are spherical.

Moreover [see below]

Thm [in progress]: If σ is swappable (as in earlier definition) (& has enough legs), then i_σ is spherical.

Moreover (for both thms): t' is induced by the "wrap once" functor (ccw) (follow the isotopy), & m' is induced by the monodromy.

Sphericality of:

$$\begin{tikzcd} A \arrow[r, "i_\sigma"] & B \end{tikzcd}$$

$A = \mathcal{W}(\sigma) \circlearrowleft m, m'$ $B = \mathcal{W}(M,\sigma) \circlearrowleft t, t'$

Cor: If Λ is swappable, then

$$LDGA \underset{C_*\Omega\Lambda \text{ modules}}{\simeq} \operatorname{cone}\left(C_*(\Omega\Lambda) \xrightarrow{f} \underset{t?}{C_*}(\Omega\Lambda)\right).$$

(characterizes unswappability?)

(strengthens that Orlov has an adjoint; sphericality $\Rightarrow$ gen. by two copies.)

• If $\phi : X \to X$ is Ham. diffeo fixing a closed exact $L \subset X$, then $\phi|_L$ induces id on $C_*(\Omega L)$ up to conjugation. (point: ϕ may not fix basepoint) (if $\dim L = 1, 2$, $\Rightarrow \phi|_L \simeq_{\text{isotopic}} id_L$).

• If σ is swappable in ∂M, then all monodromies induce the same functor. (b/c alg. determined by the adjunction) (if fillable, define dga & get contradiction).