S. Venkatesh, Action completed symplectic cohomology.

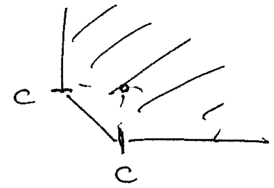
§0. Motivation.

(Ritter–Smith)

$M = \mathrm{Tot}(\mathcal{O}(-1) \to \mathbb{P}^1)$

$\updownarrow$

$(M^\vee, W)$

$\Delta_M =$ 

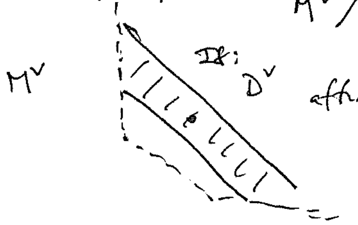
Λ : Novikov field / $\mathbb{K}$

$\mathrm{val}: \Lambda^* \to \mathbb{R}$

$$M^\vee = \{ (\zeta_1, \zeta_2) \in (\Lambda^*)^2 \mid (\mathrm{val}\,\zeta_1, \mathrm{val}\,\zeta_2) \in \mathring{\Delta}_M \}$$

$W: M^\vee \to \Lambda$ w/ crit W lies above (c,c). $\quad W = \zeta_1 + \zeta_2 + T^{-c}\zeta_1\zeta_2$.

(Ritter): $SH^*(M) \cong \mathcal{O}_{M^\vee}/(\partial_i W) \cong \Lambda$.



affinoid subdomain ~~...~~

expect also $\mathcal{O}_{D^\vee}/\partial_i W|_{D^\vee} \cong \begin{cases} \Lambda & D^\vee \supset \text{fiber above } (c,c) \\ 0 & \text{otherwise.} \end{cases}$

$\quad ?? \;\; /\!/ \mathbb{Q}.$

(on A-side).

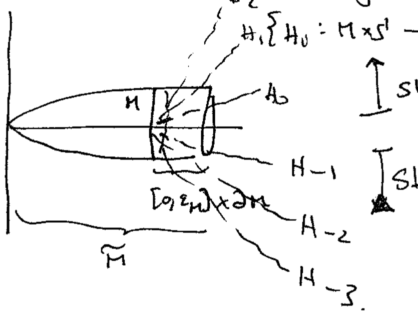
Today: focus on Liouville cobordisms w/ monotone fillings.

Plan ① define a candidate for ?

② examine its behavior on our toy case.

③ Generalizations..

① (M,ω) monotone symplectic, convex boundary.



$\{H_i : M \times S^1 \to \mathbb{R}\}_{i \in \mathbb{Z}}$ w/ H_i C^2 small on M.

& have increasing $\to \infty$ slopes on collar

$\uparrow SH^*$

$\downarrow SH_*$ $\quad \mathcal{P}(H_i)$: contractible orbits.

Choose cappings $\tilde{x}$ for each $x \in \mathcal{P}(H_i)$

on $\Gamma \subseteq \Lambda$ finite part, (polynomials / finite sums in Λ)

can define

$$\mathcal{A}_{H_i}(T^\alpha x) = \alpha - \int_D \tilde{x}^* \omega + \int_0^1 H(x(t))\,dt.$$

&

$$CF^*_a(H_i) := \Big\{ \sum_{\text{finite}} c_j T^{\alpha_j} x_j \;\Big|\; \mathcal{A}_{H_i}(T^{\alpha_j} x_j) > a \Big\} \text{ subcomplex} \quad (c_j \in \mathbb{K},\ x_j \in \mathcal{P}(H_i)),$$

$$CF^*_{(a,b)}(H_i) = CF^*_a(H_i)/CF^*_b(H_i) \circlearrowleft \partial^{\text{Floer}}. \quad (\text{w/ coeffs. in } \Gamma).$$

not a Λ module, but is a Λ_0 module. $\quad (\Lambda_0 := \{\zeta \in \Lambda \mid \mathrm{val}(\zeta) \geq 0\}.)$

Have a bidirected system:

$$\begin{tikzcd} & CF^*_{(a,b)}(H_i) \arrow[dl] \arrow[dr, bend left] & \\ CF^*_{(a,b')}(H_i) & & CF^*_{(a',b)}(H_i) \end{tikzcd} \qquad (*)$$

$b' < b$ $\qquad$ $a' < a$.

& define $SC^*_{(a,b)}(M) := \operatorname{hocolim}_{i \geq 0} CF^*_{(a,b)}(H_i)$. $(*)$ induces a bidirected system on these groups.

and $\widehat{SC}^*(M) := \varinjlim_a \varprojlim_b SC^*_{(a,b)}(M) \underset{\text{homology}}{\rightsquigarrow} \widehat{SH}^*(M)$.

↑ these two limits can be switched.

Analogously, define

$$\widehat{SC}_*(M) := \varinjlim_a \varprojlim_b \operatorname{holim}_{i<0} CF^*_{(a,b)}(H_i)$$

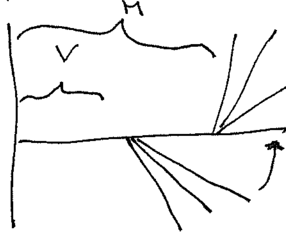
P.D. $\rightsquigarrow$ $S\widehat{H}_*(M)$ iso to the "usual notion".

Let $(W, \omega = d\theta)$ be a Liouville cobordism w/ a monotone filling V.
& let $M := V \amalg W$.

Cieliebak–Oancea define $SH^*(W)$, (when V is exact),
suggests that $S\widehat{H}^*(W)$ should be the cone of a map

$$\widehat{SC}_*(V) \longrightarrow \widehat{SC}^*(M).$$

If W is a trivial cobordism:



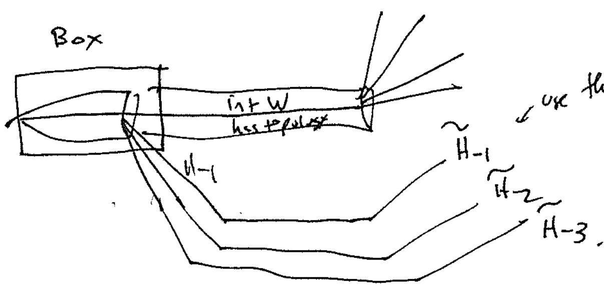
↑ c^{-1} continuation map goes here

In other words:

$$\begin{tikzcd} \widehat{SC}_*(V) \arrow[r, "\hat{v}"] \arrow[d] & \widehat{SC}^*(M) \\ CF^*(H_{-1}) \arrow[r, "c^{-1}"] & CF^*(H_0) \arrow[u] \end{tikzcd},$$

& $\widehat{SC}^*(W) := \operatorname{Cone}(\hat{v})$.

If W is not trivial:



use these Hams, but want generators in Box..

extensions

Lemma: I can choose nice $\{\tilde{H}_i\}$, J s.t. the orbits appear in Box generate a subcomplex of $\operatorname{holim}_{i\to 0} CF^*_{(a,b)}(\tilde{H}_i)$.

$\rightsquigarrow \widehat{SC}^*(W)$.

Rmk: If W is trivial: $(W \subseteq$ Liouville domain$)$.

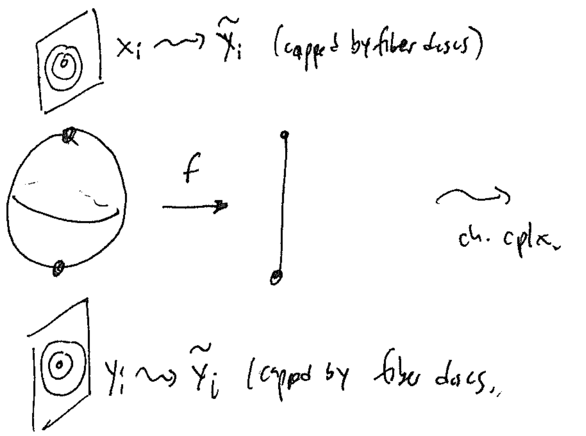
[Cieliebak - Frauenfelder - Oancea]: $SH^*(W) \cong RFH(\partial M)$.

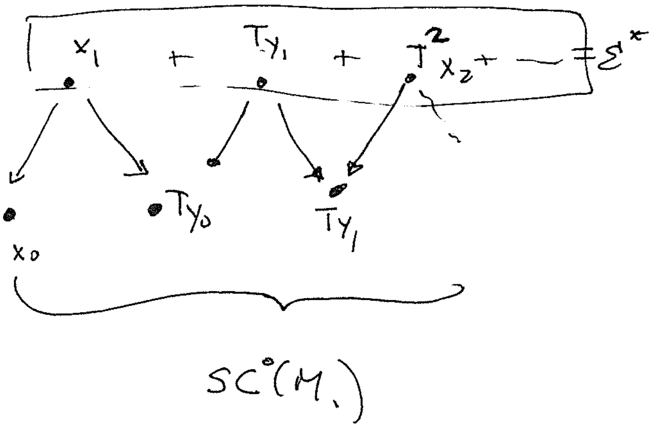
We expect a result in this flavor when $V = M$. (in particular, in completed theories, length of W matters, so want $W =$ a zero length cobordism).

§ Toy case: $M = \mathrm{Tot}(\mathcal{O}(-1) \to \mathbb{P}^1)$, $\cancel{\Omega = d((1+\pi r^2)\alpha)}$ connection 1-form. ω normalized s.t. $\int_{\mathbb{P}^1} \omega = 1$.

Albers–Kang: RFH (sphere bundles of radius $< \frac{1}{\sqrt{\pi}}$) $= 0$.

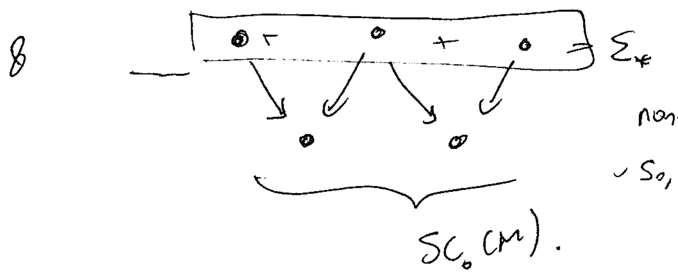
Claim this also true for $\widehat{SH}^*$ (annulus bdles w/ max radius $< \frac{1}{\sqrt{\pi}}$).





the infinite sum Σ^* kills x_0, if it exists.

Claim: Σ^* exists in $\widehat{SC}^*(\underbrace{D_R}_{\text{disc bdle of radius } R}) \iff R < \frac{1}{\sqrt{\pi}}$. (in mirror, $c = 1$ in this case!)



non-triviality depends on Σ_* existing.

So, in sum $\widehat{SH}^*(D_R) = \begin{cases} \Lambda & R \geq \frac{1}{\sqrt{\pi}} \\ 0 & \text{else} \end{cases}$ short.

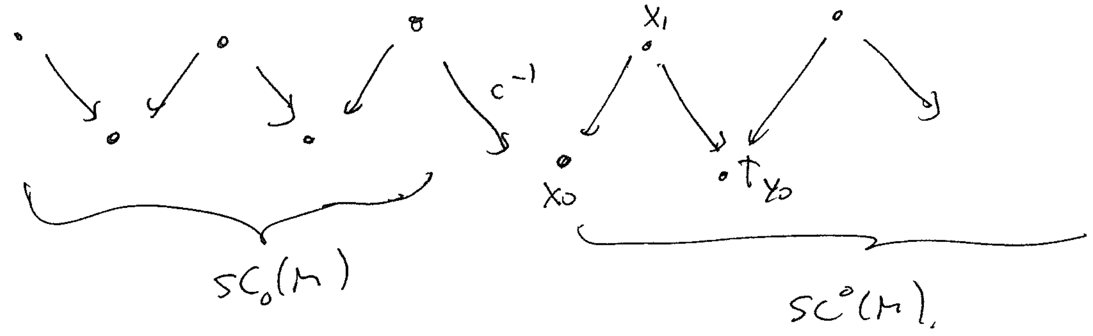
$\widehat{SH}_*(D_R) = \begin{cases} \Lambda & R > \frac{1}{\sqrt{\pi}} \\ 0 & \text{else} \end{cases}$

(Note: [Ritter-Smith]: $\mathcal{WF}(M)$ is split-gen. by by a Lag'n living in the sphere bundle of radius $\frac{1}{\sqrt{\pi}}$).

Now, also

$$\widehat{SH}^*(A_{[R_1,R_2]}) \cong \begin{cases} \Lambda & R_1 \le \overset{\frac{1}{\sqrt{\pi}}}{\overset{\shortparallel}{R}} \le R_2 \\ 0 & \text{else} \end{cases}$$

follows by analyzing



(also have non-vanishing thms for ~~[illegible]~~ cobordisms if they contain Floer-essential monotone Lag'ns. :

Thm: (V): $W \subseteq M$ monotone filling. If $L \subseteq W$ is a compact, oriented, monotone Lag'n s.t.

Liouville cobordism

$HF^*(L, E_\gamma) \neq 0$, then $\widehat{SH}^*(W) \neq 0$.

any local sys.

(pf uses an 'action-completed' open-closed map.) .

~~[illegible]~~ : (Q: what's the open-string analogue of the analogue of SH^- & HH_* / HH^- ?)